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The following papers were read:

Physiology. — "*Analysis of the curves obtained with the string galvanometer. Mass and tension of the quartz wire and resistance to the motion of the string.*" By Prof. W. EINTHOVEN. (Sequel to former communications on the string galvanometer).

(Communicated in the meeting of April 22, 1905).

CONTENTS.

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1. *Introduction.*

When recording the movements of the quartz wire in the string galvanometer with the object of becoming acquainted with various irregular oscillations of electric tension or current, we may in many cases see in the directly recorded curve an accurate or nearly accurate image of the oscillations sought. These cases are met with if the deviations of the quartz wire are aperiodic and quicker than the oscillations to be recorded. As an instance we mention here the galvanometric tracing of the human electrocardiogram, the various tops of which are reproduced in the exact proportions by the recorded curve, as was shown on a former occasion [1]).

When very rapid oscillations are to be recorded, it is necessary to make the deflections of the galvanometer also of very short duration, which can easily be done by giving a great tension to the string. In doing this, the movements of the string must in some way be damped, however, in order to avoid oscillations. The condenser method can render good service here [2]), especially if it is applied in such a way that the motion of the string is nearly brought to the limit of aperiodicity.

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1) Vide: these "Proceedings" 6, p. 107, 1903.
2) Vide: these "Proceedings" 7, p. 315, 1904.

We must remark, however, that the galvanometer loses in sensitiveness when its deflections become quicker. If the time of a deflection which is always kept at the limit of aperiodicity is reduced a times, the sensitiveness diminishes a^2 times.

Now it sometimes happens that very rapid alternations of current must be measured, which are so small, that they require a sensitive position of the galvanometer in order to be perceptible and measurable and in these cases the condenser method can no longer be applied.

If under these circumstances the deflections of the galvanometer are less rapid than the oscillations of the current which must be studied, the curve, recorded by the quartz wire, will no longer render these oscillations accurately. We must then, as has also been done with many capillary-electrometric curves, calculate from the properties of the instrument used and the data of the directly recorded curve, the value of the real oscillations. This calculation may lead to the construction of a new curve, which now in all particulars is the correct expression of the oscillations.

The manner in which capillary-electrometric curves may be calculated, is known [1]). Do the curves, recorded by the string galvanometer, fulfil the same conditions and is it possible to apply the *same* calculation to them?

The answer to this question must in general be negative. We shall try in this paper to analyse the curves of the string galvanometer and to show that they admit of equally accurate constructions as the capillary-electrometric curves, but that the manner in which they are calculated is different.

In this connection some properties of the galvanometer, more especially the tension and mass of the string and the resistance to the motion of the string must be discussed. We shall try to express the value of these three quantities in absolute measure, by which a clearer light will be shed on their significance for the practicability of the instrument generally and for the modifications which might advantageously be made in order to render the galvanometer serviceable for special purposes.

2. *The principles of the method.*

In the analysis of the curves, recorded by the string galvanometer we start from the assumption that the resistance offered by the air

[1]) For literature on this subject vide: PFLÜGER's Archiv f. d. ges. Physiol. Bd. 99, p. 472, 1903.

to the motion of the quartz wire, increases proportionally to the velocity of motion of the wire.

This assumption, which is justified on theoretical grounds, will later appear to be also practically proved by the results of analysis.

If we assume it to be true, we must also expect the motion of the string to be completely determined by the laws that govern the motion of an oscillating body with electromagnetic damping. For electromagnetic damping, as in our case the air-damping, develops a resistance, increasing proportionally to the velocity of the moving body. In the string galvanometer both electromagnetic and air-damping occur. They can be expressed in the same units and their combined effect is always equal to the effect of either of them, if only in the latter case the amount is made equal to the sum of the two, taken separately. In other words: if r_a represents the resistance through air-damping [1]) and r_b the resistance through electromagnetic damping, we may reckon for the total damping resistance:

$$r = r_a + r_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The laws of motion of an oscillating body with electromagnetic damping are e. g. extensively dealt with in KOHLRAUSCH' textbook [2]). They agree with the laws determining the charge or discharge of a condenser through a conductor having resistance and self-induction, which are found in various textbooks on electricity [3]). The formulae given in the above-mentioned works form the foundation of our further calculations.

If we denote by:

m the virtual mass of the image of the string,

r the virtual resistance, damping the motion of the string,

c the sensitiveness of the galvanometer for constant currents,

two conditions may present themselves. In the first the motion is oscillatory and

$$r < \sqrt{\frac{4m}{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in the second condition the motion is aperiodic and

$$r > \sqrt{\frac{4m}{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

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[1]) Besides the air-damping and the ordinary electromagnetic damping still other, very feeble, damping influences are active, which may however be neglected here. They will be dealt with in chapter 7.

[2]) Lehrbuch der praktischen Physik, S. 448. 1901.

[3]) Vide e.g. J. A. FLEMING, The alternate current transformer. London. I, p. 370, 1890.

The units in which m, r and c are expressed and the meaning which must be attributed to the words "virtual mass of the image of the string" and "virtual resistance" are dealt with in the following considerations on

3. *The mass of the string.*

In practice it is impossible or scarcely possible to find the mass of the string by direct weighing. For this the weight is too small. Also a calculation of the mass from the dimensions and the specific gravity of quartz and silver does not lead to the desired result, since the measurement of the diameter of the fine thread and of the thickness of the still much thinner layer of silver cannot be carried out with the necessary accuracy, the more so because the layer of silver often shows inequalities and its thickness is irregular.

So we must proceed in another way and we reach our purpose in the simplest way if we start from the "virtual mass of the image of the string". The image of the middle of the quartz thread, which is projected with great magnification, moves along a straight line over a millimetre scale each time the string deflects. Instead of this image we imagine a material point, acted on by a force which is equal to the force that moves the whole string. The acceleration experienced by the fictitious material point from this force, determines what we call the "virtual mass of the image of the string".

The unit in which this virtual mass must be expressed depends of course on the units chosen for the force and the acceleration. As the cause of the motion of the string is the electric current passed through the galvanometer, the moving force can be expressed in units of intensity of current. We choose for this purpose the micrampere.

The unit of acceleration is determined by the units chosen for the distance and the time. In accordance with the millimetre scale along which the image moves, the distance is expressed in millimetres, while it is found practical to take the millimetre also as the unit of time. For the deflection of the quartz thread is recorded on a plane which is moved along with uniform velocity perpendicularly to the direction of the motion of the image, so that in the recorded curves, from which we must always derive the data for further calculation, time is represented by a length.

The above mentioned units of force, distance and time form a system which may be called the millimetre-micrampere or $[mm—\mu A]$ system. Expressed in this system the unit of virtual mass m is the mass which experiences from a force of 1 micrampere an acceleration of 1 mm. distance per mm. of time.

The unit of virtual resistance (or better of the virtual resistance coefficient) [1]) r is the force of 1 micrampere resisting the motion of the string when the image of the string moves with a velocity of 1 mm. distance per 1 mm. of time. The unit of sensitiveness c is the sensitiveness with which a force of 1 micrampere causes a deflection of the image of 1 mm.

These units are constant as long as the field intensity H between the poles of the electromagnet, the length l of the quartz thread, the magnification b of the image of the string and the speed V of the recording plane remain unchanged.

If the string is strongly stretched and is suddenly passed by a constant current, it will perform damped oscillations. We have then the condition which we will in the first place consider more closely and which is expressed by formula (2):

$$r < \sqrt{\frac{4m}{c}} \quad \ldots\ldots\ldots \quad (2)$$

To this case also the following formulae (4) and (5) apply:

$$\tau = 2\pi\sqrt{mc}. \quad \ldots\ldots\ldots \quad (4)$$

$$T = \tau\sqrt{1+\frac{\Lambda^2}{\pi^2}}. \quad \ldots\ldots\ldots \quad (5)$$

Here means the period which we would obtain without damping, whereas T represents the real period with damping.

$$\Lambda = lgn\, k, \quad \ldots\ldots\ldots \quad (6)$$

k being the damping ratio, $k = \frac{a_1}{a_2} = \frac{a_2}{a_3} =$ etc. in which a_1, a_2, a_3 are the deflections with damping.

From the formulae (4), (5) and (6) follows:

$$m = \frac{T^2}{c\{39{,}5+4\,(lgn\, k)^2\}} \quad \ldots\ldots\ldots \quad (7)$$

As the period T is expressed in millimetres, the value of m changes if the velocity V of the recording plane is changed. In our measurements V was nearly always 500 mm. per second. Only with a few photograms the sliding frame had a different velocity, so that with these a reduction will have to be applied in order to render the values of m comparable.

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[1]) The word "resistance coefficient" expresses the meaning of r better than the word resistance. But the latter is simpler and is also used as a coefficient in the theory of electricity, where it denotes ohmic resistance. In this paper we shall repeatedly use "resistance" instead of "resistance coefficient".

In order to express m_α in the millimetre-micrampere units for a photogram of which the speed of the sliding frame is V_α mm. per second, m_β being the known value for a speed V_β, we use the relation:

$$m_\alpha = m_\beta \times \left(\frac{V_\alpha}{V_\beta}\right)^2 \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From what precedes we see that the measurements required for a calculation of the value of m when the value of V is known, are limited to:

the sensitiveness c,
the deflections a_1, a_2 . . . etc. and
the period T.

The measurement of the sensitiveness c presents no difficulties.

The intensity of the current is known from the electromotive force of the source and the resistances used. The source consisted of storage cells, the electromotive force of which remained very constant and might be put at 2 Volts, while the resistances were taken either from manganin resistance coils or from a graphite resistance of SIEMENS and HALSKE which had previously been checked.

The deflection of the quartz thread is measured on the photogram on which the network of square millimetres is found. With the magnifying glass it is easy to estimate 0.1 mm., so that a deflection of 30 mm., which is often used, is known with an accuracy of 0,3%.

If the permanent deflection is u mm., the intensity of the current $= i$ micrampere, then the sensitiveness is $c = \frac{u}{i}$ millimetres per micrampere.

The values of the oscillations a_1, a_2, etc. are, like the permanent deflection, read off directly from the network of square millimetres. But as these values are smaller than u and the absolute error in each measurement remains unchanged $= 0.1$ mm., the accuracy of the value found for $k = \frac{a_1}{a_2} = \frac{a_2}{a_3}$ . . . etc. is not great. Moreover a distinct difference is often found between $\frac{a_1}{a_2}$ and $\frac{a_2}{a_3}$, so that we are obliged to calculate a mean value, e. g. by putting $k = \sqrt{\frac{a_1}{a_3}}$. Fortunately the value of k, *as it occurs in our measurements*, has only a small influence on the final result, relatively large variations in k scarcely causing any difference in the calculated value of m.

Of the greatest importance is the measurement of the period T which in formula (7) occurs squared. The total amount of T is always very small, in some cases even less than 1 mm. so that it is desirable to carry out this measurement with an accuracy of a very small fraction of a millimeter.

For this purpose I first used an excellent astronomical instrument, destined for the investigation of celestial photograms, and which was put at my disposal by the kindness of the director of the Leyden Observatory [1]).

Later I used exclusively an ordinary microscope stand, pattern I^a of the firm CARL ZEISS. The stand is provided with the large movable cross table of the same firm. On this table a small wooden board is fastened, which follows the cross movements, being always supported by a smooth plane on which it can slide easily. On this board, which has in the middle a spacious opening, the photographic plate to be measured is placed.

As object-glass a_1 is used with a focal distance of 40 mm., while for the eye-piece that of RAMSDEN is used, also made by ZEISS, with micrometer screw and drum reading. The eye-piece is so adjusted that the direction of motion of the measuring wire coincides with the direction of the abscissae of the photogram. After the microscope has been sharply focussed on the photogram and at the same time care has been taken that the image to be measured and the image of the crosswire lie in the same optical plane, the microscope tubes are firmly screwed on to the stand by means of a clamp, which is expressly made for this purpose. The tubes must be quite immovably connected with the stand. Also the eye-piece is screwed on to the microscope tube so that it can be touched with the hand without observing the slightest displacement of the crosswire with respect to the image to be measured. For measurements requiring some accuracy these arrangements are indispensable.

With the microscope tubes pushed in, 1 scale division of the drum corresponds to a little less than 5 μ on the photogram, and since 0.1 of a scale division can easily be estimated, the error in the reading is less than 0.5 μ. It would be a mistake, however, to suppose that also the result of the whole measurement can now be known with the same accuracy. For there remain errors of a different origin.

So e. g. we start in our measurements from the shadow lines which in the photogram form the net of square millimeters, a base

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[1]) I wish to express here my best thanks to Prof. VAN DE SANDE BAKHUYZEN for his kind assistance.

which in itself is not free from errors. The ordinates of the net are recorded by a quickly revolving disk with spokes, casting 500 shadows per second on the photographic plate. The variations in the rotational speed of the disk cause only small errors. Reserving a more extensive description of this time-marker for a later occasion, we may here mention that the rotational speed is very constant and, in all probability, always agrees to 0.1 % with the pre-determined value. But we have no guarantee that the mutual distances of the spokes, although carefully marked off on the lathe, come up to the same standard.

The greatest errors are probably made by the setting of the cross-wire. To be sure, surprisingly sharp settings can be made on the fine, sharp top, representing the turning point of a movement of the string, so that a greater error than 1μ need not be made, but often it is necessary to mark the exact middle of a broader top; in which case the error becomes larger, of course. With many of our measurements of the period T the error must be estimated at 1 to 2%.

We shall now give the results of some measurements.

With a speed of the sensitive plate of $V = 100$ mm. per second, some photograms were taken of string n°. 10.

The sensitiveness was $c = 10.92$ mm. per micrampere,
the period $T = 0.43$ mm.
the damping ratio $k = 3.3$

With these data the value of m is calculated from formula (7) at 3.76×10^{-4} $[mm - \mu A]$.

For a comparison with following results this value is reduced by formula (8) for a speed of the sensitive plate of 500 mm. per second. We find then $m = 9.4 \times 10^{-3}$ $[mm - \mu A]$.

Another series of measurements with the same string gave, at any rate as far as the second figure, exactly the same result. The speed of the sensitive plate was now 500 mm. per second.

Besides c was $= 3.5$ mm. per micrampere,
T $= 1.16$ mm. and
k $= 1.83$,

from which we calculate again $m = 9.4 \times 10^{-3}$ $[mm - \mu A]$.

The agreement in the two final results is the more remarkable because of the great difference in the sensitiveness c and the period T.

String 13 is thinner than 10, having besides a smaller conductive resistance. The measurements with sudden passing of a constant current, show that with $V = 500$ and $c = 5.69$, the value of T is 1.32 and that of k 3.1. From this we calculate from formula (7) that the mass m is 6.9×10^{-3} $[mm - \mu A]$

Finally we mention still two series of measurements with a still thinner string, n° 14.

In the first photogram we find with $V = 500$, and $c = 5{,}75$ that T is 1, and $k = 3{,}8$ from which is calculated $m = 3{,}7 \times 10^{-3}\,[mm - \mu A]$.

In the second photogram V is unchanged but $c = 3{,}15$, $T = 0{,}705$ and $k = 3{,}16$.

From this we calculate $m = 3{,}5 \times 10^{-3}\,[mm - \mu A]$. So we may take for the mass of string n° 14 the mean value which is $m = 3{,}6 \times 10^{-3}\,[mm - \mu A]$.

4. *The resistance to the motion of the string.*

The resistance to the motion of the string can be determined in various ways. We shall begin with a description of the method which may be considered to give the most accurate results.

Let the quartz thread be only moderately stretched, and let it suddenly be passed by a current of constant intensity. Let the deflection of the thread be recorded on a rapidly moved sensitive plate, so that a curve is obtained, which is schematically represented in the following figure 1.

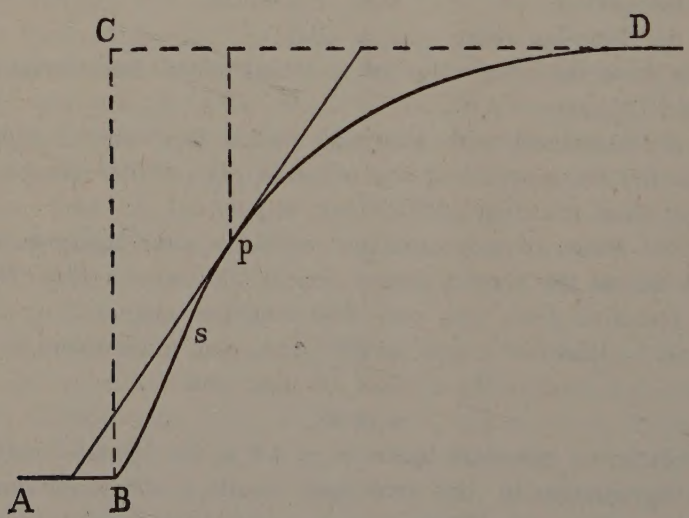

Fig. 1.

From A to B the quartz thread is in the 0-position. At B the constant current is made, by which the image of the string is moved upward until it reaches its second position of equilibrium about D.

The fundamental formula, found in the above-mentioned text-books [1]), which represents this curve, runs:

$$\frac{d^2q}{dt^2}+\frac{r}{m}\frac{dq}{dt}+\frac{1}{mc}q=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Here m, r and c have the same meaning as before, namely m the virtual mass of the image of the string, r the virtual resistance, damping the motion of the string, and c the sensitiveness of the galvanometer for constant currents. t means the time and q the distance of the image of the string from its second position of equilibrium or in other words: the distance of any point p of the curve from the line CD.

All units are expressed in the millimetre-micrampere or $[mm-\mu A]$ system.

Calling ϱ the radius of curvature in any point of the curve, we have:

$$\varrho=\pm\frac{\left\{1+\left(\frac{dq}{dt}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2q}{dt^2}} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and further putting the tangent of the angle of inclination $\frac{dq}{dt}=v$, we may write:

$$q=crv+cm\frac{(1+v^2)^{\frac{3}{2}}}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Here ϱ is positive when v increases, negative when v decreases with increase of t.

For the case that we may put $\varrho=\infty$ formula (11) simplifies into:

$$q=crv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This case must present itself somewhere in a point s of the curve. From B to s the curve is concave upward, from s to D concave downward, s itself being the point of inflection. For the point s $\varrho=\infty$, so that for this point formula (12) applies. We write it in the form

$$r=\frac{q}{cv} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In order to determine the value of the resistance r by means of this formula, the sensitiveness c of the galvanometer must be known,

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[1]) Vide KOHLRAUSCH l. c. p. 450 and FLEMING l. c. p. 368.

while two quantities must be measured with respect to the points s of the curve, viz. its distance q from the second position of equilibrium and the tangent v of the angle of inclination.

These measurements are made with the same microscope stand I^a, already mentioned in the preceding chapter, on the table of which the photographic plate can be moved by screw motions in two mutually perpendicular directions.

For object-glass a_1 is again used or another more strongly refractive lens, while for an eye-piece an arrangement is used which must be explained now.

At the outside of a HUYGENS eye-piece n°. 2 a pointer is fastened. When the eye-piece is turned round the optical axis, this pointer moves over a circular dial, screwed on to the microscope tube and divided into degrees. In this way we can read off how much the eye-piece is turned in the microscope tube. Moreover in the focal plane of the ocular lens of the eye-piece a fine crosswire is found, which is so set that the crossing point lies at the centre of the field of view. Hence, when the eye-piece is turned in the microscope tube, while the crosswire turns, the crossing point will remain immovable. The whole arrangement, of which the principal parts have been taken from the analyser of a ZEISS polarisation microscope, works very accurately and enables us to make a number of measurements in a short time. This is done in the following way:

A photographic negative or better still a diapositive is placed under the microscope, so that one of the directions of motion of the cross table coincides with the direction of the abscissae of the photogram. Next the whole microscope tube is again immovably screwed fast at such a height that the real image of the figure to be measured lies in the same plane with the wires of the eye-piece. In this way one can by means of the screws of the cross table easily and quickly cause the image of any point of the photogram to coincide with the crossing point of the eye-piece.

One begins with placing an absciss in the crossing point, and then turns the eye-piece so that one of the two crossed wires, e. g. wire A coincides with this absciss. Then an arbitrary point P of the curve is placed at the crossing point. The eye-piece is now turned again in such a way that the wire A forms a tangent line to the curve at P. The angle through which the eye-piece must be turned in order to get from the first into the second position, is the angle of inclination of the curve at P. It is read on the graduated circle in whole degrees, tenths being estimated. So a tangent can be drawn at any point of the curve after it has been brought to the crossing

point of the ocular wires, and the angle of inclination of the curve at this point can be measured. The method which also deserves recommendation for the measurement of other, especially capillary-electrometric curves, leaves little to be desired as to ease and quickness. Its accuracy can be judged from the following results.

In the reticular scales, which together with the movements of the quartz thread are photographed on the plates, the ordinates are not perfectly perpendicular to the abscissae, which must be ascribed to the circumstance that the image of the slit, formed by the cylindrical lens on the photographic plate, is not perfectly perpendicular to the direction of motion of the sliding frame. I did not carry out my original intention of correcting this error by a calculation, since the deviation only amounts to 0°.3.

Besides we can, when measuring angles of inclination less than 45°, take an absciss as base, when above 45° an ordinate, by which proceeding the deviation becomes of still less importance.

But from measurements of this deviation by means of the turning eye-piece, it appears that the measurements of angles can be made with errors not exceeding 0°.1. This will be seen from the following table.

TABLE I.

Number of the plate.	Direction of the abscissae.	Direction of the ordinates.	Difference in direction between abscissae ordinates.	Deviation from the average value of 89°.7.
A 22	110°.8	21°.05	89°.75	+ 0°.05
A 37	200°.6	111°.0	89°.6	— 0°.1
A 123	123°.0	33°.3	89°.7	0
A 124	122°.8	33°.0	89°.8	+ 0°.1
A 132	122°.9	33°.2	89°.7	0

No nearer explanation is wanted. In the last column but one we find the results of the measurement of the angles under which in various photographic plates the abscissae are cut by the ordinates. Taking for the average value an angle of 89°,7, we see that the greatest deviation from this value is only 0,°1.

It must be remarked, however, that the direction of the tangent in a point of the curve cannot be determined so accurately as the

direction of an absciss or an ordinate. The stronger the curvature of a curve and the more rapid the changes of curvature take place, the more uncertain the determination of the direction of the tangent at any point becomes.

The point s at which the curve under examination has no curvature, is easily found under the microscope.

A number of points in which the curve is cut by the abscissae or ordinates, are successively brought to the crossing point of the wires in the eye-piece and in all these points the angles of inclination are measured in the above described manner. It is noticed that with increase of the abscissae the angles of inclination first increase and then decrease. At the point of transition must be situated the required point at which the curvature of the curve is 0 or $\varrho = \infty$.

In this way the angle of inclination in the point sought, is directly known, while the distance of this point from the second position of equilibrium of the quartz thread can at once be read off on the reticular scale. Millimetres are directly indicated, tenths must be estimated.

A curve written by the quartz thread has in many cases a fairly considerable breadth. We can then consider it as a double curve, and carry out the measurement of the angles of inclination as well at the upper as at the lower side of the image of the quartz thread, in this way applying a control by which the accuracy of the final result is enhanced.

Before mentioning the amounts found by direct measurement, we must dwell a moment on the velocity of motion of the sliding frame. Most photograms were taken with a speed V of 500 mm. per second. In order to render values of v that are expressed in the millimetre-micrampere system, but were measured with different values of V, mutually comparable, they are all calculated for $V = 500$. For this purpose we use the formula:

$$r_\beta = r_\alpha \frac{V_\beta}{V_\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in which r_α means the virtual resistance of the quartz thread with a speed of the frame of V_α mm. per second, r_β being the corresponding value with a speed V_β.

We must further take into account the circumstance that the reticle on the photogram does not always consist of accurate squares. As was already mentioned in the preceding chapter, the rotational velocity

of the spoked disk, by which the ordinates are recorded, is very constant. The error in the absolute amount may be estimated at no more than 0,1 %. But the velocity of motion of the frame cannot be so accurately regulated. The speed of a frame, once moving, may be considered perfectly constant, but between photograms, taken at various times, the speed may differ fairly much. Hence the distances between the ordinates in one plate are somewhat greater than in another.

The distance between the abscissae is absolutely the same in all plates. We could not expect otherwise, since the abscissae are recorded by the shadow of a glass scale; mounted in a fixed position at a few millimetres, distance from the movable frame. The glass scale, containing very fine division lines, was expressly made for our purpose by the firm Zeiss.

The mutal distances of the abscissae and also of the ordinates are measured under the microscope by means of the millimetre divisions with nonius, which are found on the above described cross table. For 30 scale divisions along an ordinate we invariably find 30,1 mm. so that 1 scale division along an ordinate may always be put $= 1{,}0033$ mm.

The scale divisions along an absciss in the middle half or middle third part of the same plate are also equal, as may be seen e. g. from the following measurements of photogram *A* 34, see table II.

It must be remembered that the nonius did not enable us to read with a greater accuracy than 0,1 or 0,05 mm. so that for these comparatively rough measurements the mutual distances of the

TABLE II.

Number of the ordinate.	Reading on the millimetre scale.	Length of 10 scale divisions along an absciss in millimetres.
0	61.2	—
10	51.5	9.7
20	41.9	9.6
30	32.2	9.7
40	22.5	9.7
50	12.8	9.7
60	3.15	9.65

ordinates must be considered as perfectly equal. More accurate measurements have already been mentioned in the preceding chapter and need not be discussed again.

From the table follows that in the examined photogram 1 scale division along an absciss $=$ 0.9675 mm. Since the length of a scale division along an ordinate has already been given in millimetres we can directly express a scale division along an absciss in scale divisions along an ordinate. Calling this value d, we find in photogram A 34 $d = 0{,}964$. In other photograms we found values for d, varying between 1,033 and 0,901.

If d deviates somewhat considerably from unity, the value of the tangent of the angle of inclination will have to be corrected. If the tangent of the measured angle of inclination is v, we shall have to take the product vd instead of v in the computation of r.

Formula (13) is then replaced by

$$r = \frac{q}{cvd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We now give some results of measurements, for convenience' sake assembled in the following table III.

TABLE III.

Number of the quartz thread.	Number of the photo-gram.	q in milli-metres.	c in millime-tres per micrampere.	v	d	V in milli-metres per sec.	r_a in $[mm-\mu A]$ calculated for $V = 500$.
10	50	23.5	1093	6.02	1.032	100	0.0193
13	A 37	21.8	798	1.637	0.975	500	0.0174
14	A 123	23.6	1133	1.322	0.984	500	0.0160
14	A 124	21.6	563	3.006	0.901	500	0.0155

During the recording of the curves to which the numbers of this table refer, always a great resistance — more than 1 Megohm — was inserted in the galvanometer circuit, so that the electromagnetic damping might be neglected and the resistance r_a to the motion of the string was only caused by the air. We see that the value of r_a, which for all the curves has been calculated for $V = 500$, decreases in the order of the numbers 10, 13 and 14 of the quartz threads. In the same order also the thickness of the silvered strings diminishes. The diameter is:

for string 10 without silver 2,4 μ, silvered 3,0 μ
„ „ 13 „ „ 1.6 μ, „ 2.5 μ
„ „ 14 „ „ 1.7 μ, „ 1.9 μ

We leave unsettled what function the air-damping is of the diameter of the string, but wish to point out here particularly the influence which the damping may experience from the layer of silver on the quartz threads being more or less smooth. Especially string 14 has an almost perfectly smooth coat of silver, while the skin of string 10 shows distinctly visible inequalities.

We will now investigate whether the amount of r_a for a definite string may be regarded as constant, however much the tension of the quartz thread and thereby the velocity of its deflections may be modified.

For this it is necessary to measure r_a once more with a strongly stretched string and oscillating deflections, but the methods which are at our disposal for this purpose do not nearly furnish such accurate results as could be given above. We mention here two methods of measuring.

In the first place the same method as above may be used, but this time it is applied under unfavourable conditions. As the string makes rapid movements, the tangent of the angle of inclination becomes large, so that its amount cannot be determined with the desired accuracy. Besides the uncertainty in the value of q also increases very much.

Of the curves, by means of which in the preceding chapter the mass of the string was calculated, plate A 61 of string 13 shows the greatest value of T, viz. $T = 1{,}32$ mm. I therefore have chosen this curve for an attempt to determine the value of r by formula (15)

$$r = \frac{q}{cvd}.$$

A resistance of more than 1 Megohm was inserted in the galvanometer circuit so that the electromagnetic damping might be neglected and consequently r might be put $= r_a$. The following values were found:

$q = 9$ mM.
$c = 5{,}69$ mM. per micrampere.
$v = 88{,}1$
$d = 0.979$.

As the quartz thread with rapid motion writes a fine line on the sensitive plate, the angles of inclination can be measured with great

accuracy. An error of 0,1° is possible, however, in the measured value of the angle. And if the real angle is 0,1° greater than the value found, the tangent must be taken 104,2 instead of 88,1. So an error of 0,1° in the measurement would cause here an error of over 18 % in the final result. Putting the uncertainty in the value of q at 1,5 mm., also the probable error, caused by this alone, is found to be 17 %.

The measuring arrangement which we used, does not enable us to make the result more correct.

In the second method of determining the value of the resistance to the motion of the string with a strongly stretched, oscillating string, we make use of the following formula: [1])

$$lgn\ k = \frac{rT}{4m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in which the letters have the same meaning as before, viz.

$k =$ the damping ratio,

$r =$ the resistance sought,

$T =$ the period of the oscillatory deflections,

$m =$ the mass of the string.

We write formula (16) in the form:

$$r = \frac{4m\ lgn\ k}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and substitute for m the value which we found, according to formula (7) of the preceding chapter, to be

$$m = \frac{T^2}{c\{39.5 + 4\ (lgn\ k)^2\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We then find:

$$r = \frac{4T\ lgn\ k}{c\{39,5 + 4\ (lgn\ k)^2\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In order to obtain comparable results, we always calculate r for a speed of the sliding frame $V = 500$ mm. per second, using again formula (14) $r_\beta = r_\alpha \frac{V_\beta}{V_\alpha}$.

From the two last mentioned formulae the value of r is now calculated for four out of the five curves which in the preceding chapter served for the determination of m. During the recording of these curves a great resistance of more than 1 Megohm was each time inserted in the galvanometer circuit, so that the resistance to

[1]) Vide e.g. KOHLRAUSCH l. c. p. 448.

the motion of the string was again caused by friction of the air only. Hence we may identify r with r_a. The results of the calculation are found assembled in the following table IV.

TABLE IV.

Number of the quartz thread.	Number of the photo-gram.	lgn k.	T in milli-metres.	c in millime-tres per micramp.	V in millime-tres per sec.	r_a in $[mm—\mu A]$, calculated for $V = 500$.
10	46	(1.19)	0.43	10.92	100	(0.0207)
„	A 18	(0.604)	1.16	3.5	500	(0.0196)
13	A 61	(1.13)	1.32	5.69	500	(0.02355)
14	A 129	(1.335)	1.—	5.75	500	(0.0199)

Also these results lack the accuracy which can be obtained when the motion of the string is slow. The difficulty lies in the value of k which, as was remarked before, presents a great uncertainty.

In the following table V the average values r_a are given as they

TABLE V.

Number of the quartz thread.	r_a with feeble tension of the quartz thread.	r_a with strong tension of the quartz thread.	Ratio of r_a with strong tension to r_a with feeble tension.
10	0.0193	(0.0201)	1.04
13	0.0174	(0.0210)	1.21
14	0.0157	(0.0199)	1.27

were found by calculation according to the three methods mentioned.

We see that the value of r_a is a little greater with a greater tension of the quartz thread. From this we must infer that the resistance, caused by the friction of the air, does not increase quite proportionally to the velocity of motion of the string, but that with very rapid motions of the string the increase of the resistance of the air is a little greater.

Practically, however, this result has no influence on our further calculations. The increase in the value of r_a may be called very small. The greatest increase was in the ratio of 1 : 1,27, while

tensions of the quartz thread were used, varying from 1 to 324. That with smaller tensions of the string than were used above, the proportionality between the velocity of the string and the frictional resistance of the air, is maintained, has been sufficiently proved on a former occasion [1]). A standardising curve with a sensitiveness of 1 mm. detlection for 10^{-10} Ampere, was recorded on a photographic plate, which moved with a velocity of 10 mm. per second. After the sensitiveness had been reduced ten times and the speed of the plate had been ten times increased another standardising curve was written. This latter shows the same shape as the former, both curves can be so superposed that they coincide over nearly their whole length.

So we may say that the tensions may vary from 1 to nearly 3000, the value of r_a being only slightly altered.

Let us now consider the amount of the electromagnetic damping, expressed by r_b. The value of r_b is nothing else than the ponderomotive force which the string experiences by its motion in the magnetic field when the velocity of the image of the string is unity.

If this ponderomotive force is rendered in the $[mm—\mu A]$ system, it is expressed by a certain number of micramperes with the given length of the thread and the given intensity of the field.

The calculation can be made in the following way.

The electromotive force E, generated in a wire, which is perpendicular to the lines of force of a homogeneous magnetic field, and is displaced with a velocity of v_1 centimetres per second in a direction which at the same time is perpendicular to its length and to the lines of force, is

$$E = v_1 lH \times 10^{-8} \; Volt,$$

in which l means the length of the wire in centimetres and H the intensity of the field in C. G. S. units.

When the middle part of the string moves with a velocity v_1, the average velocity of all the parts of the string may be put $\frac{2}{\pi} v_1 = 0{,}637\, v_1$, so that for the electromotive force e, generated by a deflection of the quartz thread, we may write

$$e = 0{,}637\, v_1 lH \times 10^{-8} \; Volt \; . \; . \; . \; . \; . \; . \; (19)$$

Moreover we have

$$v_1 = v \times \frac{V}{10b}, \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (20)$$

[1]) See these "Proceedings" 6, p. 107, 1903.

in which, as above, v_1 means the velocity of the middle of the string expressed in centimetres per second, v the velocity of the image of the string, expressed in millimetres distance per millimetre of time, V the velocity of the frame in millimetres per second and b the magnification used.

Substituting in formula (19) the value v as given in formula (20), we have

$$e = \frac{0{,}637\, v\, VlH \times 10^{-9}}{b} \; Volts.$$

Calling the resistance in the galvanometer circuit w Ohms, the intensity of the current I, caused in the circuit by the motion of the quartz thread and expressed in micramperes, is

$$I = \frac{e \times 10^{6}}{w} = \frac{0{,}637\, v\, VlH \times 10^{-3}}{wb} \; micramperes \; . \; . \; . \quad (21)$$

The ponderomotive force, experienced by the string, is expressed by $I[mm—\mu A]$.

Hence if we put in formula (21) $v = 1$, I becomes $= r_b$ so that we have

$$r_b = \frac{0{,}637\; VlH \times 10^{-3}}{wb} [mm - \mu A] \; . \; . \; . \; . \quad (22)$$

From the above formula (22) it appears that, in order to calculate r_b, we must know V, l, w, b and H. Of these quantities the four first mentioned can be measured with sufficient accuracy, while on the other hand the value H of the intensity of the field presents some difficulty. Opposite the middle of the string there is an opening in the pole shoes of the electromagnet, by which in that place the intensity of the field, although not zero, yet becomes much smaller than in any other place of the slit-shaped space, in which the quartz thread moves.

Measurements of the intensity of the field in different parts of this slit-shaped space showed that in those places where the intensity is greatest it may be put at about 22500 (C. G. S.), the current of the field magnet being regulated at 2.7 Amp. With this current of the field magnet nearly all our observations were made. The average intensity of the field, calculated over the whole length of the slit, appeared to be about 10% less, i.e. about 20250 (C. G. S.). But this value for H is inadmissible since the places of least intensity are found at the middle of the slit where the lines of force exert the greatest influence on the motion of the string. For H we shall have to reckon considerably less than the average value of 20250

(C. G. S.), and since sufficiently extensive measurements of the strength of the field in all parts of the slit are wanting, H would have to be estimated approximately. Rather than doing this we shall follow an entirely different way.

It is possible, namely, to get to know r_b in a different manner. For this the resistance to the motion of the string must be measured twice, first when a very great conductive resistance is inserted in the galvanometer circuit and then after the conductive resistance in this circuit has been made as small as possible. In the first case the resistance to the motion of the string consists of air-damping only, in the second case of air-damping and electromagnetic damping combined. The difference between the two values gives us the value of the electromagnetic damping. Remembering formula (1) we write:

$$r_b = r - r_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

When the value of r_b has become known by means of r and r_a H can be calculated. For this purpose we write formula (22) in the form:

$$H = \frac{r_b b w \times 1000}{0.637\ Vl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

We have $b = 660$, $V = 500$ and $l = 12.7$, so that we may write for H:

$$H = 163.2 \times r_b w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

H was computed for string 13. Plate A 34 shows a curve, recorded, when suddenly a constant potential difference was established between the ends of the quartz-thread, the galvanometer circuit being closed by a small external resistance, the amount of which may be neglected. Analysis of the curve shows that $q = 24.8$, $c = 89.4$, $v = 0.98$ and $d = 0.964$, from which we derive by means of the formula $r = \frac{q}{cvd}$ (15) that $r = 0.0294$.

Formerly we found r_a to be 0.0174, from which follows that $r_b = 0.0120$.

The conductive resistance of the quartz-thread is $w = 9000$ Ohms, and from this we compute by means of formula (25) that $H = 17600$ [C. G. S.].

Two tables follow now. In table VI are found for three quartz-threads the data, enabling us to calculate the value of r by formula (15). The speed of the sliding frame is always 500 mM. per second, the galvanometer circuit being closed by a small external resistance, the amount of which may be neglected.

TABLE VI.

Number of the quartz-thread.	Conductive resistance w of the quartz-thread.	Number of the photo gram.	q in millimetres.	c in millimetres per micramp.	v.	d	r in [$mm—\mu A$].
10	10000	*A* 22	20.7	535	1.24	1.005	0.0312
13	9000	*A* 34	24.8	894	0.98	0.964	0.0294
14	17800	*A* 132	26.4	582	2.33	0.927	0.0210

In table VII we find the values of r_a from table III together with those of r from table VI. The difference between the two values is indicated in the last column but one by r_b; the last column gives the value of r_b as calculated by formula (22), for H the value 17600 [C. G. S.] having been taken.

The accordance between the values of these two columns is very satisfactory and may be considered as a proof of the accuracy with which in general the resistance to the motion of the string may be determined by means of the formula $r = \frac{q}{cvd}$ (15).

The ratio of the directly measured values of r_a and r is

for string 10 as 1 : 1,615
„ „ 13 „ 1 : 1,69
„ „ 14 „ 1 : 1,34

TABLE VII.

Number of the string.	w in Ohms.	r_a measured.	r measured.	r_b calculated from the measured values of r and r_a.	r_b calculated from the length of the string and the field-intensity.
10	10000	0.0193	0.0312	0.0119	0.0108
13	9000	0.0174	0.0294	0.0120	0.0120
14	17800	0.0157	0.0210	0.0053	0.0061

To conclude this chapter we give some remarks concerning the condition for which the motion of the quartz-thread just reaches the limit of aperiodicity. In this condition we have the relation :

$$c = \frac{4m}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

According to this formula one would be induced to determine the value of c from m and r, but it will appear in chapter 6 that for small tensions of the quartz-thread the virtual mass of the image of the string is not a constant value.

Hence it is not or hardly possible to derive from measurements that were performed with other tensions of the string, the value of m for the case that the limit of aperiodicity has been reached. And if m is unknown c cannot be calculated, of course.

So if one wants to know the sensitiveness for which the limit of aperiodicity is reached, one is obliged to determine this directly by experiment. The results of a number of such determinations which, as will be understood, were only made in a rough way, are found united in the following table VIII.

From the data of the preceding table and from the values of r it would be possible to calculate the values of m.

Similarly those of the time-constant[1]) $\mathbf{T}$ by the formula $\mathbf{T} = \frac{rc}{2}$.

These calculations, however, must be omitted, since c has a far too small degree of accuracy here, to attach any importance to the results.

TABLE VIII.

Number of the string.	Sensitiveness c for the limiting value of aperiodicity.	
	with air-damping only.	with air-damping and electromagnetic damping combined.
10	(120)	(50)
13	(130)	(45)
14	(115)	(55)

5. *The acceleration.*

When analysing a curve, recorded by the capillary electrometer, if we wish to know the potential difference which at a certain moment exists between the mercury and the sulphuric acid, we have, besides the properties of the instrument and the speed of the recording

[1]) See FLEMING, l.c. pp. 377 ff.

plane only to take into account the velocity of the motion of the meniscus. When analysing a curve, however, recorded by the string galvanometer, the analogous data are not sufficient in general. It will often be necessary to take into account not only the velocity but also the acceleration presented by the image of the string.

This must be ascribed to the fact that in the capillary electrometer the resistance to the motion of the meniscus is very great [1]) compared with the mass of the mercury thread, so that this mass may be neglected, when it is desired to calculate the existing potential difference from the velocity of motion, whereas with the string galvanometer the resistance to the motion of the quartz-thread is very small, and hence the mass of the thread in many cases has a distinct influence on the velocity of its deflections.

These considerations may be succinctly rendered by formula (11), already developed in the preceding chapter:

$$q = crv + cm\frac{(1+v^2)^{\frac{3}{2}}}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If r is very great compared with m the second term behind the $=$ sign may be dropped and the formula becomes

$$q = crv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This formula (12) can be applied as well for the analysis of capillary electrometric curves as for curves of the string galvanometer for which v is small and ϱ large.

On the other hand, for moderate values of v and ϱ the mass m may no longer be neglected, so that then analysis of the curve will only be possible if besides the velocity also the acceleration can be measured. This acceleration, expressed as virtual acceleration of the image of the string in millimetres distance per millimetre of time, is nothing else but $\frac{(1+v^2)^{\frac{3}{2}}}{\varrho}$.

Assuming as known the general conditions under which a curve is written by the string galvanometer, and also the distance from any point of the curve to the zero line, one will have to measure the tangent v of the angle of inclination and also ϱ the radius of curvature, in order to calculate the potential difference which existed between the ends of the quartz-thread at the moment, that the arbitrary point mentioned was recorded. Under unchanged general conditions each

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[1]) On the influence of frictional resistance on the movement of the meniscus in LIPPMANN's capillary electrometer, see these "Proceedings" II, p. 108, 1899.

point may be said to be fully characterised by its distance from the zero line and the values of v and ϱ.

The distance from the zero line may very easily be determined by the presence of the square millimetre net, while in a preceding chapter it was pointed out how v is measured. So we have only to describe the best way of ascertaining the value of the radius of curvature ϱ.

Three different methods were tried for measuring ϱ of which one only proved practicable. The other two will only be briefly mentioned. First a reduced diapositive was made photographically of a large drawing on which a number of circles with different, accurately known radii were represented. On the diapositive the radii vary systematically from 0,5 mm. to ∞. It must be so laid on the curve to be measured that one of the circles coincides with the curve in any point of this latter. By direct comparison the value of ϱ in that point will then be known.

In the second method three points of the curve are measured, situated at small but mutually equal distances. Calling k the distance of the two extreme points and p the distance of the middle point from the straight line that joins the two extreme points, the radius of curvature at the spot where the measurement is made, is

$$\varrho = \frac{k^2 + 4p^2}{8p}.$$

Here k represents the chord and p the height of the circular arc under measurement.

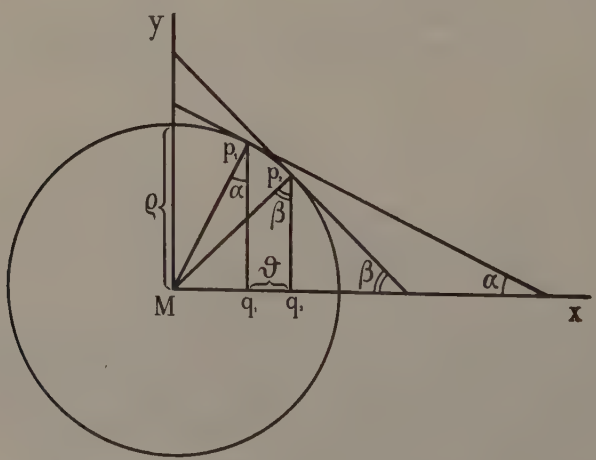

Fig. 2.

The third method, the only one that could be applied with good results, as was remarked above, consists in measuring the angles of inclination in two points of the curve, situated near each other.

Let p_1 and p_2 be two near points of a curve, the radius of curvature of which keeps the same value ϱ in all points between p_1 and p_2, the angle of inclination at p_1 being represented by α and that in p_2 by β.

MX is an abscis in the coordinate system which was recorded as a net of square millimetres together with the curve, but has been omitted in the figure, while MY, p_1q_1 and p_2q_2 are ordinates.

It is seen from the figure that

$$\sin \alpha = \frac{Mq_1}{\varrho} \text{ and } \sin \beta = \frac{Mq_2}{\varrho}.$$

Putting $Mq_2 - Mq_1 = \vartheta$ we have

$$\varrho = \frac{\vartheta}{\sin \beta - \sin \alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The value of ϑ can be read off in a simple manner on the net of square millimetres, while the angles α and β must be measured by means of the eye-piece with cross-wires. This arrangement and the accuracy that can be obtained by it, have already been dealt with in the preceding chapter; we now put the question in what cases the determination of ϱ may or may not be practically useful.

Let us once more consider formula (11)

$$q = crv + cm \frac{(1+v^2)^{\frac{3}{2}}}{\varrho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad . \quad (11)$$

this time as the expression of a curve, representing the damped oscillations of a strongly stretched quartz-thread. For each reversing point the value of v must be put $= 0$. Hence for a reversing point the formula becomes

$$q = \frac{cm}{\varrho}$$

$$m = \frac{q}{c} \varrho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

in which the sensitiveness c is an accurately known quantity. So it would only be necessary to determine q and ϱ in order to be able to calculate at once a value for m from every reversing point.

But here the practical difficulty lies in the quick variations which ϱ shows already for moderate values of q. The time ϑ has now to be taken so small that it can no longer be measured with sufficient

accuracy, at any rate with our measuring arrangement, when the microscope has been mounted with the cross-wire eye-piece. And hence ϱ itself becomes inaccurately known.

So we conclude that measuring ϱ has no practical value when we want to know the value of m for a strongly stretched oscillating string. Moreover this value has already been determined for this case in a satisfactory manner by the method described in chapter 3.

But the measurement of ϱ does obtain practical value when we want to know the virtual mass of the image of a string, which has written a curve with a feeble or moderate tension of the quartz-thread [1]). When analysing different curves it is not sufficient to use the calculated real mass of the quartz-thread, since, as has already been mentioned and will still more clearly appear in the following chapter, the virtual mass of the image of the string is very considerably modified by changes in the tension of the quartz-thread.

We finally remark that when the velocity v is great, also the angles α and β become large, by which the difference of the sines diminishes for the same difference of the angles. This causes a diminution of the accuracy with which ϱ can be known.

Also when ϱ becomes very great the determination loses in accuracy, since then for an equal value of v the difference between $\sin \alpha$ and $\sin \beta$ greatly diminishes. But this drawback is of no practical consequence, as in the analysis of a curve the value of ϱ, as soon as it gets beyond a certain limit, may be put ∞ without a large error.

6. *Analysis of some curves.*

We give in this chapter the results of the analysis of some curves, written, when a known, constant potential difference was suddenly established between the ends of the quartz-thread.

The first of the curves to be dealt with was recorded with a rather feeble tension of the quartz-thread, i.e. with a rather sensitive position of the galvanometer. 1 mm. ordinate $= 1{,}87 \times 10^{-9}$ Amp. or the sensitiveness $c = 535$. The speed of the sensitive plate is $V = 500$ mm. per sec., so the value of 1 mm. abscis $= 2\,\sigma$.

We call $t = 0$ the moment when the electric current is started. Now the angles of inclination of the curve are measured at $t = 1\,\sigma$, $2\,\sigma$, $3\,\sigma$, etc. In the following table IX, in the first column the values of t are expressed in thousandths of a second and in the

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[1]) How a separate calculation of ρ can be avoided here, will appear in the following chapter.

TABLE IX. (String 10, Plate *A* 22).

1	2	3	4	5	6
t in thousandths of a second.	v	$r \, c \, v$	q in millimetres. Measured.	q in millimetres. Calculated.	Difference between the measured and calculated value of q in mm.
0	—	—	24.6	—	—
1	0.762	12.7	24.2	17.8	— 6.4
2	0.863	14.5	23 8	21.7	— 2.1
3	1.000	16.7	23.3	23.4	0.1
4	1.083	18.1	22.7	23 8	1.1
6	1.238	20.7	21.4	20.7	— 0.7
—	(1.236)	(20.7)	(20.7)	(20.7)	(0)
8	1.235	20.7	20.1	20.7	0.6
10	1.139	19 1	18.9	18.4	— 0.5
12	1.103	18.5	17.7	17.9	0.2
14	1.028	17.2	16.6	16.4	— 0.2
16	0.945	16.0	15.5	15 5	0
18	0.926	15.5	14.5	15.0	0.5
20	0.856	14.2	13.6	13.6	0
22	0.798	13.4	12.7	13.0	0.3
24	0.770	12.9	11.9	—	1.0
26	0.705	11.8	11.1	—	0.7
98	0.680	11.4	10.4	—	1.0
30	0.615	10.3	9.7	—	0.6
32	0.572	9.6	9.1	—	0.5
34	0.541	9.1	8.5	—	0.6
36	0.501	8.4	8.0	—	0.4
38	0.481	8.1	7.5	—	0.6
40	0.451	7.6	7.1	—	0.5
42	0.423	7.1	6.5	—	0.6
44	0.398	6.7	6.2	—	0.5
46	0.368	6.2	5.8	—	0.4
48	0.361	6.0	5.4	—	0.6
50	0.350	5.9	5.1	—	0.8
52	0.311	5.2	4.7	—	0.5
54	0.303	5.1	4.4	—	0.7
56	0.279	4.7	4.1	—	0.6
66	0.181	3.0	2.8	—	0.2
76	0.135	2.3	2.0	—	0.3
86	0.0945	1.6	1.5	—	0.1
96	0.0682	1.1	1.0	—	0.1

second column the values of the tangent v of the angles of inclination existing at these moments.

In the third column the values of the product rcv are given, calculated in the following manner.

If of the first part of the curve the concave side is directed upward, the concave part of the second half is directed downward. At the point of inflection $\varrho = \infty$.

Here by formula (13)

$$r = \frac{q}{cv} \quad \ldots \ldots \ldots \quad (13)$$

or

$$rc = \frac{q}{v},$$

so that from the values here given for q and v, the value of rc can be calculated. For any other point of the curve the constant value rc is then multiplied by the value of v for that point.

In the fourth column the values of q, i.e. the distances of the image of the string from the second position of equilibrium, are given as the results of direct measurements.

In the fifth column the values of q are given as calculated from the formula

$$q = crv + cm\frac{tg\beta - tg\alpha}{\vartheta}, \quad \ldots \ldots \quad (28a)$$

while in the sixth column the differences between the measured and calculated values of q are given.

The above formula (28a) requires some explanation.

It is obtained by replacing in formula (11) the value $\frac{(1+v^2)^{\frac{3}{2}}}{\varrho}$ by $\frac{tg\beta - tg\alpha}{\vartheta}$.

As was remarked above $\frac{(1+v^2)^{\frac{3}{2}}}{\varrho}$ or $\frac{d^2q}{dt^2}$ is nothing else but the expression for the acceleration. Since we used as the only method for measuring ϱ the measurement of two angles α and β, see fig. 2, we can also, these angles being known, find an approximate expression for the acceleration by means of their tangents.

The velocity at the point p_1, fig. 2, is given by $tg\alpha$, at the point p_2 by $tg\beta$. The difference in velocity is $tg\beta - tg\alpha$. Assuming the acceleration to be constant during the time ϑ, it is expressed by

$$\frac{tg\beta - tg\alpha}{\vartheta}.$$

By using formula (28a) instead of (11) we considerably simplify the calculations. In the first place it becomes unnecessary to look up the sines of α and β, while the tangents of these angles are already known, since they were wanted for the determination of crv. And then it is not necessary to calculate the values of $(1 + v^2)^{\frac{3}{2}}$.

The data, serving for the determination of the acceleration, have been collected in table. X. One also finds in the sixth column the

TABLE X (string 10, Plate A 22).

1	2	3	4	5	6	7	8
t in thousandths of a second.	ϑ in millimetres.	$tg\alpha$	$tg\beta$	$\frac{tg\beta - tg\alpha}{\vartheta}$	Difference between crv and the measured values of q in millimetres.	$cm\frac{tg\beta - tg\alpha}{\vartheta}$ in millimetres.	Algebraic sum of the values of the two preceding columns in millimetres.
1	1	(0.696)[1]	0.836	0.167	— 11.5	5.1	— 6.4
2	1	0.762	1.000	0.238	— 9.3	7.2	— 2.1
3	1	0.863	1.083	0.220	— 6.6	6.7	0.1
4	2	0.863	1.238	0.187	— 4.6	5.7	1.1
6	—	—	—	(0)	— 0.7	(0)	— 0.7
8	—	—	—	(0)	0.6	(0)	0.6
10	2	1.235	1.103	— 0.066	0.2	— 0.7	— 0.5
12	2	1.139	1.028	— 0.055	0.8	— 0.6	0.2
14	2	1.103	0.945	— 0.079	0.6	— 0.8	— 0.2
16	2	1.028	0.926	— 0.051	0.5	— 0.5	0
18	2	0.945	0.856	— 0.045	1.0	— 0.5	0.5
20	2	0.926	0.798	— 0.064	0.6	— 0.6	0
22	2	0.856	0.770	— 0.043	0.7	— 0.4	0.3

[1]) α has been calculated here by the formula $\alpha = 2\gamma - \beta$, in which γ represents the angle of inclination of the curve at the time $t = 1$.

difference between rcv and the measured value of q, in the seventh column the value of $c\,m\,\frac{tg\beta - tg\alpha}{\vartheta}$ and in the last column the algebraic sum of the values of the sixth and seventh columns.

If our measurements had an absolute precision the values of the last column would all be $= 0$.

In the calculation of the tables IX and X, the correction has been neglected which must be applied if a scale division along an abscis is not equal to a scale division along an ordinate. Thus we assumed that the net of square millimetres consists of real squares or in other words that $d = 1$. This has no influence on the calculated value of rcv, since the correction of r compensates that of v; but it has an influence on the values of $\frac{tg\beta - tg\alpha}{\vartheta}$. But the differences are not of such an order of magnitude that the correction would be necessary; the general form of the curve remains unaltered.

The results of the measurements and calculations which are given in figures in the preceding table IX, can be illustrated by means of a diagram. In fig. 3 which corresponds to table IX, the net of square millimetres is represented at about twice its natural size. A scale division along an abscis $= 2\,\sigma$; a scale division along an ordinate $= 1{,}87 \times 10^{-9}$ Ampère.

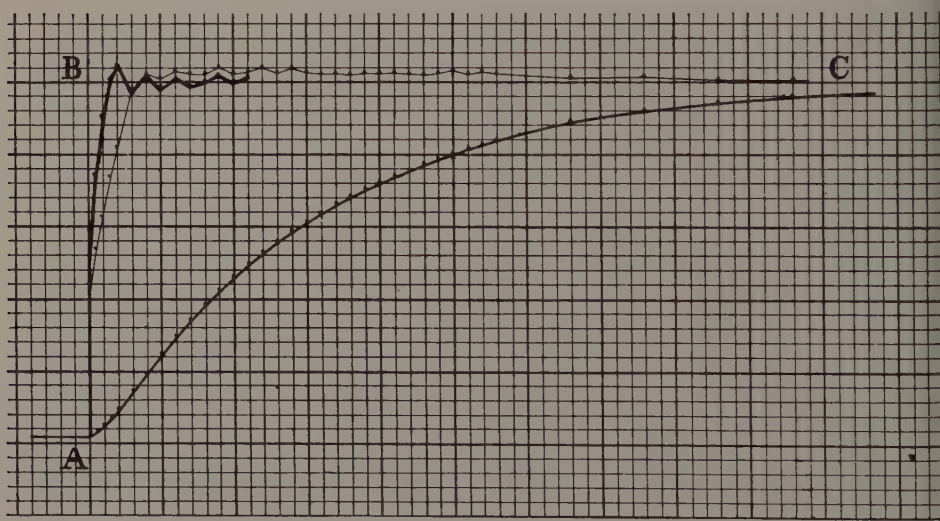

Fig. 3

String N°. 10, Photogram A 22, Tables IX and X.
Abscis 1 scale division $= 2\tau$; ordinate 1 scale division $= 1{,}87 \times 10^{-9}$ Amp.

The regularly bent line of medium thickness represents the recorded curve. At the moment $t = 0$ the constant current is made. In the case of an ideal, absolutely accurate analysis we should find two straight lines, one of which would rise vertically from A to B, while the other from B to C would be horizontal. The results of the actual analysis according to column 5 of table IX are represented by the thick line, while the thin line represents the values of rcv according to column 3 of this table.

For the virtual mass m two different values have been used; for the first 4 thousandths of a second m has been put 0.0567 which is 6 times the amount found in chapter 3. At $t = 6\,\sigma$ and $t = 8\,\sigma$, m has no influence, since at these times ϱ may be put $= \infty$. Beginning with $t = 10\,\sigma$, m has again been reckoned but this time with a value 0,0187 which is twice the value of chapter 3.

If a single value is given to m the results become much less satisfactory and the question arises whether the whole analysis must not be considered worthless now that it appears to be impossible to assume a constant value for m.

In reply to this we remark in the first place that, as will presently appear, the variation in the value of m is only of account with a great sensitiveness of the galvanometer, i. e. with a feeble tension of the quartz thread. Besides, even with the most sensitive position of the galvanometer still an important part of the analysis can be usefully applied. For practically a curve, such as is obtained e. g. in many electrophysiological investigations, will consist of parts of various curvatures and will always show a number of points for which ϱ may be put $= \infty$ or the acceleration may be put $= 0$. In all these points m need not be taken into account. Since r can be measured with great accuracy the analysis is in all respects satisfactory here.

Moreover the analysis can be applied wherever the curvature and at the same time the angle of inclination are not too great — in our case already in all points that are recorded later than 0.004 second after the starting of the current — as will appear from table IX and figure 3. For each definite tension of the quartz thread a definite value for m be taken.

The reason why in general m is represented by another value for different tensions of the quartz thread will be discussed in chapter 8. But here we must ask why m can also vary when the sensitiveness of the galvanometer and together with it the tension of the quartz thread remain unchanged. For an explanation of this unforeseen and somewhat disappointing phenomenon we have in the first

place sought for errors in the measurements, which might be occasioned by the not sharply defined edge of the quartz thread being

TABLE XI (string 14, Plate *A* 132).

1	2	3	4	5	6
t in thousandths of a second.	v	$r\,c\,v$	q in millimetres. Measured.	q in millimetres. Calculated	Difference between the measured and calculated values of q in millim.
0	—	—	32.7	—	—
1	1.664	18.8	31.9	27.0	— 4.9
2	1.872	21.2	31.0	29.9	— 1.1
3	2.069	23.4	30.0	30.8	0.8
4	2 215	25.0	29.2	29.7	0.5
5	2.290	25.9	28.0	28.4	0.4
6	2 332	26.4	26.9	26.4	— 0 5
—	(2 332)	(26.4)	(26.4)	(26.4)	(0)
7	2.332	26.4	25.9	26.4	0.5
8	2.204	25.0	25.0	24.5	— 0.5
10	2.087	23.6	23.0	23.0	0
12	1.881	21.3	21.1	20.6	— 0.5
14	1.753	19.8	19.5	19.2	— 0.3
16	1.613	18.3	17.9	17.9	0
18	1.511	17.1	16.5	16.7	0.2
20	1.418	16.0	15.1	15.5	0.4
22	1.280	14.5	13.9	14.0	0.1
24	1.179	13.3	12.8	13.0	0.2
26	1.111	12.6	11.7	12.3	0.6
28	1.032	11.7	10.7	11.4	0.7
30	0.942	10.7	9.9	—	0.8
32	0.848	9.6	9.0	—	0.6
34	0.821	9.3	8.2	—	1.1
36	0.751	8.5	7.5	—	1.0
38	0.676	7.7	6.9	—	0.8
40	0.635	7.2	6.3	—	0.9
42	0.563	6.4	5.8	—	0.6
44	0.504	5.7	5.2	—	0.5
46	0.456	5.2	4.8	—	0.4
48	0.423	4.8	4.4	—	0.4
50	0.382	4.3	4.0	—	0.3
52	0.350	4.0	3.7	—	0.3
56	0.304	3.4	3.1	—	0.3
60	0.254	2.9	2.6	—	0 3
70	0.151	1.7	1.7	—	0

TABLE XII (string 14, plate *A* 132.)

1	2	3	4	5	6	7	8
t in thousandths of a second.	ϑ in millimetres.	$tg\alpha$	$tg\beta$	$\frac{tg\beta - tg\alpha}{\vartheta}$	Difference between $r\,c\,v$ and the measured values of q in millimetres.	$cm\ \frac{tg\beta - tg\alpha}{\vartheta}$ in millimetres.	Algebraic sum of the values of the two preceding columns in millimetres.
1	1	(1.488)[1]	1.872	0.384	— 13.1	8.2	— 4.9
2	1	1.664	2.069	0.405	— 9.8	8.7	— 1.1
3	1	1.872	2.215	0.343	— 6.6	7.4	0.8
4	1	2.069	2.290	0.221	— 4.2	4.7	0.5
5	1	2.215	2.332	0.117	— 2.1	2.5	0.4
6	—	—	—	(0)	— 0 5	(0)	— 0.5
7	—	—	—	(0)	0.5	(0)	0.5
8	2	2.332	2.087	— 0.122	0	— 0.5	— 0.5
10	2	2.204	1.881	— 0.161	0.6	— 0.6	0
12	2	2.087	1.753	— 0.167	0.2	— 0.7	— 0.5
14	2	1.881	1.613	— 0.134	0.3	— 0.6	— 0.3
16	2	1.753	1.511	— 0.121	0.4	— 0.4	0
18	2	1.613	1.418	— 0.097	0.6	— 0.4	0.2
20	2	1.511	1.280	— 0.115	0.9	— 0.5	0.4
22	2	1.418	1.179	— 0.119	0.6	— 0.5	0.1
24	2	1.280	1.111	— 0.084	0.5	— 0.3	0.2
26	2	1.179	1.032	— 0.073	0.9	— 0.3	0.6
28	2	1.111	0.942	— 0.084	1.0	— 0.3	0.7

photographically distorted where the curve bends. But the errors so caused are far too small to explain the matter; moreover they are

[1]) α has again been calculated here by the formula $\alpha = 2\gamma - \beta$, see the note at the foot of table X.

to a great extent eliminated if the measurements at the lower side of the string are controlled by measurements at the upper side.

The most likely explanation has to be sought, in my opinion, in the lack of homogeneity of the magnetic field. The middle part of the quartz thread is placed between the objectives of the microscopes, where the magnetic field is only very feeble compared with the field in which the other parts of the thread are placed. The ponderomotive force which causes the quartz thread to deflect, when it is passed by a current, is consequently smaller in the middle of the string than at the two ends. These latter, so to say, draw aside the middle part and so it can be understood that with feeble tension of the quartz thread the displacement of the middle part lags.

The stronger the tension of the quartz thread the more regularly it will move over its whole length. So we may expect that for a less sensitive position of the galvanometer the values which we must take for m will be more equal among each other.

In tables XI and XII and figure 4 belonging to them, we will first give the analysis of a curve written by string n°. 14. The tables

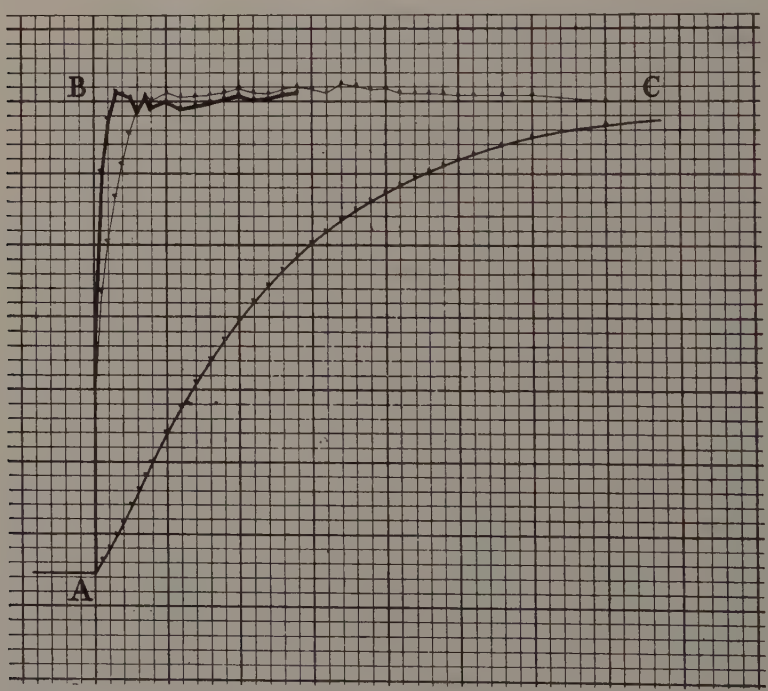

Fig. 4.

String N°. 14, photogram *A* 132, Tables XI and XII.
Absciss 1 scale division $= 2\tau$, ordinate 1 scale division $= 1{,}72 \times 10^{-9}$ Amp.

or diagram require no nearer explanation since they are in every respect analogous to those of string n°. 10 that were discussed above. We have here again $V = 500$ hence 1 mm. abscis $= 2\sigma$. Further $c = 582$, so 1 mm. ordinate $= 1.72 \times 10^{-9}$ Amp. The value of m has been put during the first 5 thousandths of a second at 0.037 which is rather more than 10 times the value found in chapter 3. At $t = 6\sigma$ and $t = 7\sigma$, m has no influence. Beginning with $t = 8\sigma$, m has been taken into account again, this time with a value of 0.00688 or rather more than 1.9 times the value of chapter 3.

Some other curves, also recorded with a sensitive position of the galvanometer show after analysis diagrams that agree completely with the two above described diagrams 3 and 4. So they require no nearer elucidation here.

We will not omit, however, to give the results of the analysis of a curve recorded with a less sensitive position of the galvanometer. The numbers are collected in tables XIII and XIV, which like the corresponding diagram 5 have been arranged in the same manner as the preceding tables and diagram. They represent a curve written by string 14, with 1 megohm in the galvanometer circuit. Here we have $c = 115{,}2$, hence ordinate 1 mm. $= 8.67 \times 10^{-9}$ Amp., while abscis 1 mm. is again $= 2\,\sigma$.

The value of m can be kept constant here at a value which is 1.45 times greater than the value which would hold for strong tension of the string.

We see that most of the examined points are calculated with an error, smaller than 1 mm. and that the correction is already pretty accurate after $1\,\sigma$. After $1\,\sigma$ the error amounts to 1.4 on a total deflection of 30.6 mm., i. e. 4.6 %. This proves that by means of analysis of the curve with a sensitiveness of the galvanometer $c = 115\,2$, for which 1 mm. deflection corresponds to a current of 8.67×10^{-9} Amp., the real intensities of the current can be known beginning at $1\,\sigma$ after starting the current, and then progressing $0.5\,\sigma$ each time.

In all probability these times can still be materially shortened when the speed of the photographic plate is increased. With the curves mentioned in this paper a speed of the sliding frame of 500 mm. per second has chiefly been used, but it is evident that with improved mechanical appliances it will be possible to attain greater speeds. We have lately succeeded in obtaining very regular speeds of 1 M. per second.

At the end of this chapter we remind the reader that an analysis of the curves is only necessary when it is desired to measure very

TABLE XIII (string 14 plate *A* 125).

1	2	3	4	5	6
t in thousandths of a second.	v	$r c v$	q in millimetres. Measured.	q in millimetres. Calculated.	Difference between the measured and calculated values of q in millim.
0	—	—	30.6	—	—
0.5	10.78	18.2	28.4	22.2	— 6.2
1	12.71	21.45	25.6	24.2	— 1.4
1.5	13.15	22.2	22.2	(22.2)	(0)
2	12.71	21.45	19.3	18.7	— 0.6
2.5	10.89	18.4	17.0	15.2	— 1.8
3	10.02	16.9	14.2	14.3	0.1
3 5	8.71	14.7	12.0	11.7	— 0.3
4	7.50	12.65	10.2	10.1	— 0.1
5	5.700	9.6	7.3	7.8	0.5
6	4.504	7.6	4.9	6.0	1.1
7	3.078	5.2	3.3	3.9	0.6
8	2.251	3.8	2.1	3.0	0.9
9	1.688	2.85	1.2	2.1	0.9
10	1.163	2.0	0.6	1.4	0.8
11	0.740	1.2	0.2	0.8	0.6

feeble currents in very short times. As soon as the galvanometer may be less sensitive, by applying a method of damping which was formerly described, curves may be obtained, directly recording the accurate intensity of the current in less than 1σ.

TABLE XIV (string 14, plate *A* 125).

1	2	3	4	5	6	7	8
t in thousandths of a second.	$\mathfrak{s}$ in millimetres.	$tg\alpha$	$tg\beta$	$\frac{tg\beta - tg\alpha}{\mathfrak{s}}$	Difference between rcv and the measured values of q in millimetres.	$cm\ \frac{tg\beta - tg\alpha}{\mathfrak{s}}$ in millimetres.	Algebraic sum of the two preceding columns in millimetres.
0.5	0.5	(9.36)[1]	12.71	6.70	— 10.2	4.0	— 6.2
1	0.5	10.78	13.15	4.74	— 4.2	4.8	— 1.4
1.5	—	—	—	(0)	(0)	(0)	(0)
2	0.5	13.15	10.89	— 4.52	2.1	— 2.7	— 0.6
2.5	0.5	12.71	10.02	— 5.38	1.4	— 3.2	— 1.8
3	0.5	10.89	8.71	— 4.36	2.7	— 2.6	0.1
3.5	0.5	10.02	7.50	— 5.04	2.7	— 3.0	— 0.3
4	1	10.02	5.700	— 4.32	2.5	— 2.6	— 0.1
5	1	7.50	4.504	— 3.00	2.3	— 1.8	0.5
6	1	5.700	3.078	— 2.622	2.7	— 1.6	1.1
7	1	4.504	2.251	— 2.253	1.9	— 1.3	0.6
8	1	3.078	1.688	— 1.390	1.7	— 0.8	0.9
9	1	2.251	1.163	— 1.088	1.6	— 0.7	0.9
10	1	1.688	0.740	— 0.948	1.4	— 0.6	0.8
11	1	1.163	(0.439)[2]	— 0.724	1.0	— 0 4	0.6

[1]) Calculated, see the note at the foot of table X.

[2]) β has here been calculated in the same way as the first α of the table.

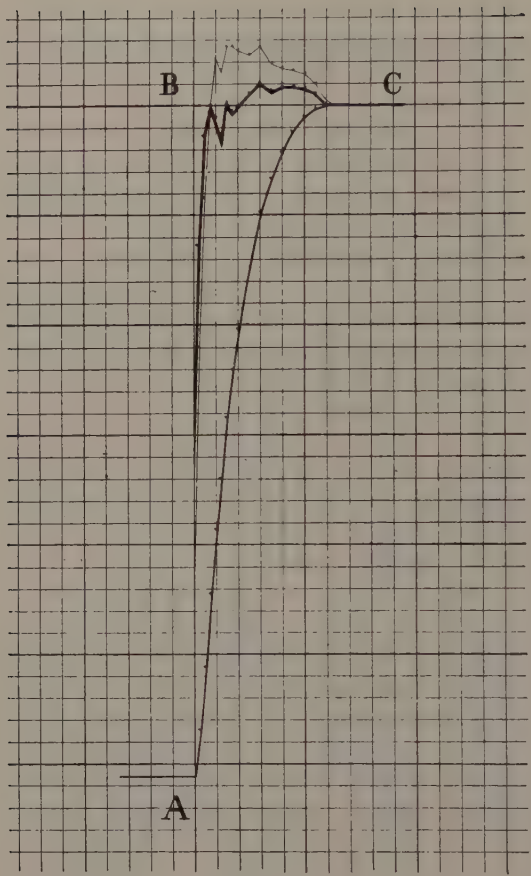

Fig. 5.

String N°. 14, photogram A 125, Tables XIII and XIV.
Absciss 1 scale division $= 2\sigma$, ordinate 1 scale division $= 8{,}67 \times 10^{-9}$ Amp.

7. *Absolute measures of the mass of the string and the resistance to the motion of the string.*

As soon as m, the virtual mass of the image of the string is known in millimetre-micrampere units, it is not difficult to calculate the real mass of the string in grammes. In order to do this we must first in formula (7) express the values of T and c in the ordinary units of the (C.G.S.)-system and then the circumstance has to be taken into account that although the middle of the string and consequently the image of the string performs rectilinear oscillations,

yet the motion of the quartz thread as a whole has a more complicated character.

Calling m_1 the real mass of the string in grammes, T_1 the period in seconds and c_1 the sensitiveness in centimetres deflection of the middle of the string per dyne, we have

$$m_1 = m \times \left(\frac{T_1}{T}\right)^2 \times \frac{c}{c_1} \times \frac{\pi^2}{8} \quad . \quad . \quad . \quad . \quad (29)$$

The factor $\frac{\pi^2}{8}$ is introduced by the circumstance we mentioned already above. The method by which the amount of this factor is calculated will be given later, after the tension of the string will have been dealt with. We now proceed to a nearer discussion of the values of $\frac{T_1}{T}$ and $\frac{c}{c_1}$.

T represents the time in millimetres while the velocity of motion of the sliding frame is V mm per second. The time in seconds is consequently

$$T_1 = \frac{T}{V}$$

or

$$\frac{T_1}{T} = \frac{1}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In order to determine the value of $\frac{c}{c_1}$, we must take into account the magnification used, b, the strength of the magnetic field H and the length of the string l.

H is expressed in (C.G.S.) units and l in centimetres.

c, as was mentioned before, is the sensitiveness, expressed in millimetres deflection of the image of the string per micrampere, c_1 being the sensitiveness, expressed in centimetres deflection of the middle of the string itself per dyne.

The force deflecting the quartz thread when a current of 1 micrampere is passed, is $\frac{Hl}{10^7}$ dynes. Hence

$$c = c_1 \times \frac{Hl}{10^7} \times 10b,$$

or

$$\frac{c}{c_1} = \frac{Hlb}{10^6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

By means of the formulae (29), (30) and (31) we can now express m_1 in m and find:

$$m_1 = \frac{\pi^2}{8}\, m \times \frac{Hlb}{10^6\, V^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The accuracy with which m_1 can be calculated in grammes depends of course in the first place upon the accuracy with which m is known in $[mm—\mu A)$ units and further upon the accuracy of the values of H, l, b and V. The latter quantity occurs in formula (32) squared and so would have a preponderant importance. But the time-recording arrangement which we used, works, as we saw formerly, with so great an accuracy that we may neglect errors in the value of V.

Also l and b can be measured with sufficient accuracy, while for H the value has been taken which we found in chapter IV, namely 17600 [C. G. S.].

The error in the absolute value of m_1 I estimate at a few percent.

In formula (32) we have.

$$H = 17600,$$
$$l = 12{,}7,$$
$$b = 660,$$
$$V = 500,$$

from which follows that $m_1 = 7{,}28 \times 10^{-4}\, m$ . . . . . (33)

In chapter III we found that:

for string N°. 10 $m = 9{,}4 \times 10^{-3}$ $[mm — \mu A]$
„ „ „ 13 $m = 6{,}9 \times 10^{-3}$ „ „
„ „ „ 14 $m = 3{,}6 \times 10^{-3}$ „ „

By formula (33) we calculate from this the mass of the strings in absolute measure:

for string N°. 10 $m_1 = 6{,}85 \times 10^{-6}$ gram.
„ „ „ 13 $m_1 = 5{,}02 \times 10^{-6}$ „
„ „ „ 14 $m_1 = 2{,}62 \times 10^{-6}$ „

We passingly remark that for recording sounds we use a very light short string: a 2,5 cM. long, 1 μ thick quartz thread, of which the weight may be estimated at about $1{,}5 \times 10^{-7}$ grammes.

From the length l, the diameter d of the naked quartz thread and the specific gravity of quartz s, the weight of the quartz may be calculated as

$$g = \frac{d^2}{4}\, \pi\, ls.$$

This weight is only inaccurately known on account of the uncertainty in d. But combined with the value of m_1 it may serve us to obtain a rough idea of the relative weights of quartz and silver in the string. Calculated in this manner we find this ratio

string 10	:	1	quartz to	3.5	silver
„ 13	„	1	„ „	6.4	„
„ 14	„	1	„ „	2.4	„

We now proceed to express the resistance to the motion of the string in absolute measure. According to the definition formerly given r is the virtual resistance to the motion of the string in micramperes, when the image of the string moves with a velocity of 1 mm distance per 1 mm time.

We call r' the resistance to the motion of the string in dynes when the middle of the string moves with a velocity of 1 cM. per second.

The above mentioned unit r refers to a strength of field H, a length of the quartz thread l, a magnification b and a speed of the writing plane V.

Since the force of 1 μA is equal to $\frac{Hl}{10^7}$ dynes, we may write:

$$r' = r \times \frac{Hl}{10^7} \times \frac{10\,b}{V}$$

or

$$r' = r \times \frac{Hlb}{10^6 V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Substituting in this the above values for H, l, b and V, we get

$$r' = 0{,}295\, r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It is unnecessary to give here the absolute measures of the electromagnetic damping. These were discussed in chapter IV where they served us for accurately determining the value of H.

On the other hand the absolute measures of the air-damping r'_a may find a place here.

In chapter IV the air-damping was found

for string N°. 10 $r_a = 0{,}0193$ $[mm - \mu A]$
„ „ „ 13 $r_a = 0{,}0174$ „ „
„ „ „ 14 $r_a = 0{,}0157$ „ „ .

By formula (35) we calculate from this

for string N°. 10 $r'_a = 0{,}00569$ dynes.
„ „ „ 13 $r'_a = 0{,}00513$ „
„ „ „ 14 $r'_a = 0{,}00463$ „

It would be desirable to compare these values with those that might be calculated by means of the kinetic theory of gases. But then we ought to bear in mind that we have combined in r_a' other causes of damping besides the air-damping.

These causes are threefold:

1 If the magnetic field is non-homogeneous, there may arise vortex currents in the layer of silver during a deflection of the quartz thread.

2. If the string is para- or diamagnetic, it may by its motion induce currents in the iron of the pole shoes.

3. Also a non-magnetic string will induce a movement of electricity in the pole shoes when it is passed by a current and moves.

But all three causes are so small that they may probably be neglected compared with the air-damping.

8. *The tension of the quartz thread.*

S.

In order to calculate the tension of the quartz thread under various circumstances, we begin with assuming a special case, namely that the thread is strongly stretched and is placed over its entire length in a homogeneous magnetic field. A constant current passed through the galvanometer causes a permanent deflection of the thread which assumes the shape of a catenary.

Calling u_1 the deflection of the middle of the thread and i_1 the ponderomotive force experienced by the thread, the tension is:

$$S = \frac{i_1 l}{8 u_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Here S and i_1 are expressed in dynes, while the deflection u_1 and the length l are given in centimetres.

Now

$$\frac{u_1}{i_1} = c_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in which c_1, denotes the sensitiveness of the galvanometer, as was already mentioned with formula (29), expressed in centimetres deflection of the middle of the thread per dyne.

From formulae (36) and (37) follows that

$$S = \frac{l}{8 c_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

And from (31) and (38) we derive that the tension is

$$S = \frac{H l^2 b}{8 \times 10^6 c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Substituting again for H, l and b their values, viz. $H = 17600$, $l = 12.7$ and $b = 660$, we find $S = 234 \times \frac{1}{c}$ dynes, which result, calculated in grammes, gives for the value of the tension

$$S = 0.239 \times \frac{1}{c} \text{ grammes} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

We see from these formulae that the tension is inversely proportional to the sensitiveness.

For a sensitiveness $c = 1$ the tension would be 239 milligrammes. Assuming with THRELFALL [1]) that a thin quartz thread has a tensile strength of 100 kilogrammes per mM² section, when using a string of 2.39 μ^2 section or 1.75 μ diameter, the sensitiveness of the galvanometer may be diminished to $c = 1$, i.e. to 1 mm. deflection for 1 micrampere without the thread breaking. The strongest tension we applied with string n°. 14 corresponds to a sensitiveness of 1 mm. deflection for 3×10^{-7} Amp., hence $c = 3.3$, while the diameter of the string amounts to 1.7 μ.

From these data it appears that the maximum tension used by us is still 3 times smaller than the tensile strength of the string. We remark here that this limit has only been calculated for the quartz, while the silver which might also contribute something to the strength has been left out of account.

String n°. 10, when uncoated, has a diameter of 2.4 μ. From this we calculate that it will bear such a tension that the sensitiveness of the galvanometer is reduced to a minimum of $c_{min} = 0.529$. The maximum of the practically applicable sensitiveness is $c_{max} = 10^5$. The ratio $\frac{c_{max}}{c_{min}} = 1.89 \times 10^5$ indicates the possible variation in sensitiveness, which may undoubtedly be called enormous.

The value found for c_{max} gives rise to some remarks about the corresponding tension S_{min}. According to formula (40) we should find for $c = 10^5$ a tension of 2.39×10^{-6} grammes, an absurd value, since the weight of string n°. 10, here used as an example, amounts to 6.85×10^{-6} grammes, i. e. nearly three times more, and since of course the tension in a vertically stretched string cannot possibly be less than its weight. But this absurd result is easily explained by remembering that formula (40) only holds if the quartz thread is strongly stretched and so behaves as a string, which was premised in the calculation of the tension.

From the results obtained we must conclude that with feeble tension the quartz thread no longer moves like a string. There are sufficient data to prove that the motion of the quartz thread does not even completely agree with the vibration of a string when the

1) Philosoph. Magaz. Vol. 30 (5), p. 99. 1890.

sensitiveness has been diminished to $c = 100$ and hence the tension is over 300 times the proper weight of the thread.

We must not lose sight of the fact that besides the stretching arrangement there are various forces which act on the quartz thread. If the thread is so thin that its elasticity may be neglected while only gravity has to be reckoned with, it will assume, when entirely relaxed, the shape of a catenary. If it is paramagnetic it will be bent towards one of the poles in the strong magnetic field which is never perfectly homogeneous. And if its elasticity may not be neglected, it will assume shapes, determined by the position and the direction of the extremities at the places of attachment, while also a slight torsion about the longitudinal axis may make itself felt.

When the tension of the thread is gradually increased by screwing the upper end of the thread upwards, it is easy to observe with the microscope the moment at which the thread is pulled straight.

Before the thread is straight its middle will be displaced nearly in a horizontal plane when the stretching arrangement is screwed up. After the thread has been pulled straight the middle is displaced upwards on account of the elongation of the thread, this displacement being half that of the extremity of the thread. The thread once having been pulled straight, already a small increase in tension will force it to move like a stretched string.

The results of the calculation about the tension of the quartz thread are in agreement with the relation existing between this latter and the distance of the two extremities of the thread. Of most of the quartz threads it might be assumed that they were just on the verge of being stretched when their sensitiveness corresponded to about 1 mm. deflection for 10^{-8} Amp., or $c = 100$. In this position very small changes in the mutual distance of the extremities cause already great differences in the sensitiveness. But when the mutual distance of the extremities has once been so far shortened or lengthened that the sensitiveness has thereby become either considerably increased, e. g. to 1 mm. deflection for 10^{-10} Amp, or considerably reduced, e. g. to 1 mm. deflection for 10^{-7} Amp., conditions are altered. Then great displacements can be given to the upper end of the quartz thread by means of the stretching arrangement, causing only relatively small changes in sensitiveness.

Further the sensitiveness of the thread, once stretched, shows an increase which is inversely proportional to the increase in length. So e. g. with string n°. 13 the sensitiveness will be diminished for an extension of $100\,\mu$ from $c = 100$ to $c = 10$, and for another $100\,\mu$ from $c = 10$ to $c = 5$. The sensitiveness is inversely proportional

to the tension so that we may say that the increments of the tension are proportional to the increments of the length, which must be expected for a stretched elastic thread. For increments in length of ratios 0 : 1 : 2 the increments in tension are as $\left(\frac{1}{100}\right) : \frac{1}{10} : \frac{1}{5}$ or as $\left(\frac{1}{10}\right) : 1 : 2$.

We now proceed to derive the factor $\frac{\pi^2}{8}$ in formula (29) and suppose again that the string is strongly stretched and is placed over its whole length in a homogeneous magnetic field.

According to the laws, obeyed by the vibrations of a string, we have :

$$S = \frac{4lm_1}{\tau_1^2},$$

in which τ_1 denotes the period in seconds if no damping were present, while, as was mentioned before, S represents the tension in dynes, l the length in centimetres and m_1 the real mass of the string in grammes.

Now by formula (38) we also have $S = \frac{l}{8c_1}$ so that we may write

$$\frac{4lm_1}{\tau_1^2} = \frac{l}{8c_1}$$

or

$$m_1 = \frac{\tau^2_1}{32c_1} \quad \ldots\ldots\ldots \quad (41)$$

From formula (4) we know that $\tau = 2\pi\sqrt{mc}$ or $m = \frac{\tau^2}{4\pi^2 c}$, from which follows, having regard to formula (41), that

$$\frac{m_1}{m} = \left(\frac{\tau_1}{\tau}\right)^2 \times \frac{c}{c_1} \times \frac{\pi^2}{8},$$

and since $\frac{\tau_1}{\tau} = \frac{T_1}{T}$ we may also write

$$m_1 = m \times \left(\frac{T_1}{T}\right)^2 \times \frac{c}{c_1} \times \frac{\pi^2}{8}.$$

This formula is identical with formula (29) which proves that the factor sought by us is indeed $\frac{\pi^2}{8}$.

We make a short digression here about the calculation of the sought factor for the case that the motion of the quartz thread deviates from the vibration of a string. We shall still assume, however, that the thread is over its whole length in a homogeneous magnetic field.

In the first place it is easy to tell when the sought factor must be equal to unity. The stretched thread ought then always to move perpendicularly to its length, and ought to execute in its entirety exactly the same movements which in reality are only executed by the middle of the string.

In the second place we shall make the calculation for the case that the two halves of the thread after deflection, form the two sides of an isosceles triangle, while we assume that the movements of the middle of the thread are the same as those of the middle of a real string. The sought factor then gets the value $^3/_2$ and is found in the following manner.

The kinetic energy of the thread is calculated while it is in the phase of its quickest motion. Let the velocity of the middle of the thread then be v_1 and let the mass xm_1 be distributed evenly over the whole length of the thread. Under these circumstances and with the assumption that the two halves of the thread always remain straight lines, the kinetic energy is

$$E = \frac{xm_1v_1^2}{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Let the first mentioned imaginary thread for which we found the factor 1 have a mass m_1 and let it execute the same movements as the middle of the last mentioned thread. Then its energy in the same phase of motion will be

$$E_1 = \frac{m_1v_1^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Call the permanent deflection u_1 and the total ponderomotive force k, then the work done by the ponderomotive force when a deflection has been made, is

in the first case $E_1 = k \times u_1$,

in the second case $E = k \times \frac{1}{2} u_1$,

so that $E_1 = 2E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$

From formulae (42), (43) and (44) now follows that

$$\frac{xm_1v_1^2}{3} = \frac{m_1v_1^2}{2}$$

and consequently that $x = \frac{3}{2}$.

9. *The practicability of the string galvanometer for special purposes.*

For some purposes it may be desirable in order to judge of the

practicability of a galvanometer to know its normal sensitiveness. This is calculated by the formula [1])

$$E_s = \frac{a}{10t^2 Iv\sqrt{w}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where E_s denotes the normal sensitiveness,

a the deflection in millimetres,
t the period of a whole oscillation ($\sim$) in seconds, calculated for undamped vibrations, [2])
I the current in amperes,
v the microscopic magnification, and
w the internal conductive resistance in ohms.

On account of an interesting paper by WALTER P. WHITE [3]) we remark that the quantity *normal sensitiveness* is sharply defined in a formula and hence need not give rise to misunderstanding. The quantity mentioned may be very useful for forming an idea of changes which may eventually be made in an existing galvanometer or may advantageously be applied in the construction of a new instrument. In the normal sensitiveness one has a valuable, important datum about a galvanometer, but it is obvious that the practicability of the instrument is still far from being determined by it.

For when judging of the practicability a number of other properties play an important part, as e.g. the internal resistance that can be reached in practice, the amount of the damping, the constancy of the zero point, the proportionality of the deflections to the currents, etc.

The normal sensitiveness is

for thread 10 . . . . . 10^6,
„ „ 13 . . . . . $1{,}4 \times 10^6$,
„ „ 14 . . . . . 2×10^6,
„ „ 20 (passingly mentioned in chapter 7) $2{,}1 \times 10^6$.

If we could succeed in making an aluminium wire of 1 μ diameter and 12.7 cm. length, i.e. of the same length as each of the three first quartz threads, we should obtain a galvanometer of which the

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[1]) See formula (5) in Ann. d. Physik. 12. p. 1063. 1903.

[2]) In Ann. d. Phys. l.c. we denoted by t the duration of a complete oscillation. This qualification implies that our formulae only hold for periodic motion. We think it desirable to mention expressly here, that the period t must be calculated for undamped oscillations.

[3]) WALTER P. WHITE. Sensitive moving coil galvanometers. The Physical Review vol. 19, n°. 5, p. 305. 1904.

internal resistance would be 5180 ohms and the normal sensitiveness 35×10^6.

We will now discuss some other conditions by which the practicability of the galvanometer for various purposes is determined.

Galvanometric methods may be divided into:

I. Those in which an oscillating deflection is required.

II. Those in which the deflection must preferably be aperiodic.

The first methods are subdivided into I *A*, those with a slow period as in an ordinary ballistic galvanometer, and I *B*, those with a quick period, as in the optical telephone of MAX WIEN and the vibration galvanometer of RUBENS.

The methods mentioned under I *A* are applied for the measurement of capacities and of small times by POUILLET's method, in general always when small quantities of electricity have to be measured.

Now the properties of the string galvanometer enable us to measure these small quantities of electricity also with an aperiodic deflection. If the electric current is only of short enough duration the deflection of the string is in fact proportional to the quantity of electricity passed.

For the smallest quantity of electricity which can still be demonstrated we found on a former occasion [1]) as the result of a rough calculation 5×10^{-12} ampere-seconds, corresponding to the charge of a sphere of 4.5 cm. radius at a potential of 1 volt. This calculation was for a deflection of 0.1 mm. of string n°. 10. For string n°. 18 the actual measurement was made. The sensitiveness appeared to be a little greater still: 1 mm. deflection for 4×10^{-11} coulombs, so that with this thread a quantity of 4×10^{-12} coulombs can be demonstrated.

But the sensitiveness for small quantities of electricity would still be considerably increased if the damping of the motion of the string could be removed or diminished, e.g. by enclosing the string in a vacuum. We should then obtain a slowly oscillating quartz thread which would be thousands of times more sensitive than the most sensitive ballistic galvanometers now existing.

I *B*. The string galvanometer can very well serve as an optical telephone or as a vibration galvanometer and so advantageously replace the telephone as well for measurements of self-induction as of electrolytic resistances.

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[1]) See these "Proceedings" 6, p. 707, 1904.

For the first purpose I used it with good results [1]) when instead of the silvered quartz thread a thin metal wire was stretched between the poles of the electromagnet. It appeared to be very easy to make the period of the vibrations of the string agree with that of the alternating currents used. In a few seconds one has increased or diminished the tension of the string accurately to the desired amount and for my purpose neither the sensitiveness nor the certainty of the reading left anything to be desired.

If it should be necessary to increase the sensitiveness, a vacuum could be applied, by which one would be enabled to obtain less damped vibrations even of the lightest quartz thread. It must be remarked that a vacuum is not always necessary for obtaining little damped vibrations, especially when alternating currents of very short period, e. g. of 0.001 second and less are used. For the greater the tension of the quartz thread, the smaller the damping ratio becomes.

II. The methods in which the deflection of the galvanometer must preferably be aperiodic are distinguished as II A, those with slow, and II *B*, those with quick deflection.

II. A. Of those with slow deflection we choose two examples : the measurement of currents with great external resistance, such as is applied for examining insulation resistances and the measurement of currents with small external resistance such as thermo-currents.

In both these measurements deflections of long duration, e. g. of 10 to 20 seconds can be used with good result. Here the normal sensitiveness of the quartz thread in the galvanometer, as it is now mounted with strong air-damping in the Leyden laboratory, no longer plays a part. Under these circumstances the mass has only a small influence on the movement of the thread and the velocity of the deflection is chiefly determined by the amount of the damping. This latter only depends on the friction of the air when insulation resistances are measured.

If by applying a vacuum the movement of the quartz thread could be brought near the limit of aperiodicity and at the same time the deflection could be made slow by sufficiently relaxing the tension of the thread, an instrument would be obtained by which insulation resistances could be measured, many thousands of times greater than is now possible with the most sensitive galvanometers.

In the measurement of thermo-currents some of the good points of

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[1]) Ueber Nervenreizung durch frequente Wechselströme. PFLÜGER's Archiv f. d. ges. Physiol. 82, S. 101, 1900. See also "Onderzoekingen" Physiol. Laborat. Leyden. 2nd series IV and V.

the string galvanometer come out least. Besides the difficulty of the air-damping one meets here that of the electromagnetic damping, which soon becomes very considerable.

With unchanged strength of the field the electromagnetic damping is inversely proportional to the ohmic resistance of the circuit. With thread n° 10 the air-damping and electromagnetic damping are as 1 : 0.6, the ohmic resistance in the closed circuit being 10.000 ohms. When measuring an insulation resistance the electromagnetic damping vanishes and the thread will want about 15 seconds for a deflection, when the sensitiveness is regulated at 1 mm for 10^{-11} amp. When measuring a thermo-current, for which the external resistance in the circuit may be neglected and only the galvanometric resistance of 10.000 ohms has to be reckoned, for the same sensitiveness the duration of a deflection will be 1.6 times greater i.e. 24 seconds. Putting the condition that the duration of a deflection shall not exceed 15 seconds, one has to be contented with a 1.6 times less sensitiveness and obtains 1 mm deflection for 1.6×10^{-11} amp. or for $1{,}6 \times 10^{-7}$ volt.

Since a deflection of 0.1 mm can still be observed in practice, the now existing galvanometer will be able to show a P. D. of 1.6×10^{-8} volt when thermo-currents are measured.

The application of a vacuum would only little increase the sensitiveness for small potential differences, and would not reduce the minimum to more than 0.6×10^{-8} volt. Also using a quartz thread with smaller resistance will only cause little change in this sensitiveness. If the ohmic resistance becomes n times less the smallest observable difference of potential will become $\left(\frac{1}{n} + 0.6\right) \times 10^{-8}$ volt.

But there are two means for increasing the sensitiveness for a potential difference, which must be mentioned. They consist in making the strength of the field smaller and in shortening the quartz thread.

We suppose the string to be placed in a vacuum so that the damping of its motion is only caused by electromagnetic influences. It is further assumed that the deflections are aperiodic and so slow that the influence of the mass of the string on its velocity of motion may be neglected. If under these conditions the strength of the field is reduced a times, and at the same time the tension of the string a^2 times, for an equal duration of a deflection the sensitiveness will be a times increased.

But it is easy to show that a useful diminution of the intensity of the field cannot be driven very far. For it must be remembered

that the influence of the mass of the string on its velocity of motion can only then be neglected, if a strong damping is present.

If by diminishing the intensity of the field one goes on reducing the damping, and yet wishes to retain the aperiodicity of the deflection, one will be at last obliged to make the quartz thread, the ohmic resistance remaining the same, still lighter than it is already.

We can generalise these considerations, and at the same time calculate the obtainable maximum of sensitiveness for thermo-currents in absolute measure, if we proceed as follows.

We put the condition that the deflection of the thread shall be aperiodic and that the duration of a deflection shall not exceed a pre-determined amount, e.g. 10 seconds. The most favourable conditions are then obtained if the movement of the thread is just brought at the limit of aperiodicity.

We further assume that of damping influences only the electromagnetic damping has to be reckoned, either because the thread is in a vacuum, or because the electromagnetic damping has so increased that relatively to it the air-damping may be neglected.

At the limit of aperiodicity the formula, mentioned at the close of chapter IV, holds:

$$c = \frac{4m}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and besides

$$\mathbf{T} = \frac{2m}{r},$$

in which $\mathbf{T}$ represents the time constant [1]).

Both formulae refer to the $[mm—\mu A]$ system. Expressing r_1 in dynes, m_1 in grammes and $\mathbf{T}_1$ in seconds, we get

$$r_1 = \frac{l^2 H^2}{w} \times \frac{2}{\pi} \times 10^{-9} \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$c = \frac{m_1 l H}{r_1^2} \times \frac{32}{\pi^2} \times 10^{-6}\, b \text{ millimetres per micrampere.} \quad . \quad (47)$$

and

$$\mathbf{T}_1 = \frac{m_1}{r_1} \times \frac{16}{\pi^2} \text{ seconds} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

From formulae (46), (47) and (48) we derive that

$$c = \frac{w}{Hl} \times 10^3 \pi \times b\, \mathbf{T}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

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[1]) See FLEMING, l. c. p. p. 377 seq.

Calling c_s the sensitiveness for a potential difference, expressed in millimetres deflection per microvolt, we have

$$c = c_s w$$

from which follows, together with formula (49) that

$$c_s = \frac{1}{Hl} \times 10^3 \pi \times b \, \mathbf{T}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

We further derive from formulae (46) and (48) that

$$\mathbf{T}_1 = \frac{m_1 w}{l^2 H^2} \times \frac{8 \times 10^9}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

These last two formulae (50) and (51) furnish us with all the data for easily examining the influence of various changes in the galvanometer on its sensitiveness for thermo-currents.

In the first place we point out that making a thread thinner or thicker has no influence on the sensitiveness c_s, if only the product $m_1 w$ in formula (51) remains unaltered.

Using a metal wire, the value of $m_1 w$ remains naturally the same however the thickness of the wire may vary, if always wires of the same metal and of the same length are used. It may be advantageous to use a heavy, thick wire, since then the air-damping may be neglected, without the wire requiring to be placed in a vacuum. Also the practical difficulties of applying a very feeble tension may perhaps in this case be more easily solved by means of an elastic stretching arrangement than when a thin wire is used.

In the second place we point out that by formula (50) the sensitiveness c_s is inversely proportional to the intensity of the field and to the length of the wire.

We first give our attention to the intensity of the field and imagine a thread of constant length $l = 12.7$ cm. The question how far the intensity of the field may under these conditions be diminished, can be answered by means of formula (51).

In order to raise the sensitiveness to a maximum, the field strength must be reduced to a minimum. If $\mathbf{T}_1$ and l are constant, then according to (51) $m_1 w$ must be made a minimum. Using a thread of homogeneous material, $m_1 w$ is only determined by the nature of the material, so that the question about the minimum of H is reduced to the question for what material $m_1 w$ is a minimum. As far as I can judge this is the case for aluminium, which has for $l = 12.7$ cm. a value of $m_1 w_{Al} = 1.394 \times 10^{-3}$.

Assuming for $\mathbf{T}_1$ the value 2.5 seconds, the deflection has been nearly completed after 10 seconds. A distance of 1.85 % of the total deflection remains to be travelled through then. After 12.5 sec. this

distance has been reduced to 0.68 %, after 15 seconds to 0.25 % of the total deflection.

Now putting in formula (51) $\mathbf{T}_1 = 2.5$, $l = 12.7$ and $m_1 w$ as minimum $= 1.394 \times 10^{-3}$, the minimum of H works out at 940 (C.G.S.) By formula (50) we calculate from this the maximum of sensitiveness $c_s = 434$ mm per microvolt.

Let us now consider the shortening of l. If a limit was soon found where diminution of H ceased to be useful, this is not the case with the shortening of l, which may be pushed as far as we like as long as no practical difficulties are met with. By making l shorter e.g. a times, as well the mass as the ohmic resistance are each reduced a times. The value of $m_1 w$ thus becomes a^2 times less, so that $\mathbf{T}_1$ remains unaltered (formula 51) and the sensitiveness c_s (formula 50) becomes a times greater.

A last remark may follow about the two formulae (50) and (51). We first assume that they are both valid, and that the values of $m_1 w$, l and H have been so chosen that $\mathbf{T}_1 = 2.5$. We next assume that the mass m_1 is changed, while all the rest of the instrument, including w, remains constant, and ask how the movement of the wire is altered by this. When m_1 is increased, the motion of the wire becomes oscillatory. When m_1 is diminished the motion remains aperiodic but transgresses the limit of aperiodicity. The duration of the deflection is lengthened while the sensitiveness remains the same.

This latter case agrees with the actual conditions in the string galvanometer used by myself. The mass of the quartz thread is in reality very small. If it were $= 0$ the duration of the deflection would be exactly twice as great as when m_1 possessed the desired value. Hence there is under these circumstances an advantage in increasing the mass of the wire to a certain value. [1])

String n°. 18 has a mass and an air-damping which were not accurately measured, but which will not differ much from the corresponding values of string n°. 10. Its ohmic resistance is about 2 times smaller, however, and amounts to 5100 ohms. With a time of deflection of about $^1/_2$ minute the sensitiveness is $c_s = 20$ mm. per microvolt. If I could increase the mass of this string in a practicable manner, I should with unaltered sensitiveness bring the

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1) The time constant is doubled when $m = 0$. See FLEEMING l. c.

It may be superfluous to remark that for the measurement of insulating resistances increase of m_1 will offer the same advantages as were mentioned above for the measurement of thermo-currents.

motion at the limit of aperiodicity, and obtain a time of deflection of about 15 seconds.

We remark here that thread n°. 18 may easily be so feebly stretched that its time of deflection becomes about one minute, by which the sensitiveness is increased to $c_s = 40$ mm. per microvolt. Since, as is proved by the photograms, 0.1 mm. can still be read off, with thread 18 a P.D. of 2.5×10^{-9}volt can actually be demonstrated. Also with this feeble tension of the thread the zero point remains constant, while the image of the quartz thread remains sharp over a pretty long part of the scale. It may be considered remarkable that one should be able to displace so slowly with the greatest regularity a suspended little thread of only a few thousanths of a milligramme weight.

II *B*. We now come to the methods in which the deflection of the galvanometer must be aperiodic and at the same time quick.

These methods in the first place find an application in electrotechnics, e. g, for investigating the shape of the oscillations of potential and current obtained by means of dynamos, interrupters, induction apparatus, etc. For these purposes the oscillograph is already used with good results, which instrument possesses a considerably smaller sensitiveness than the string galvanometer, but yet can be of excellent service in the measurement of stronger currents.

In the second and for our purpose most important place the methods mentioned under II *B* find their application in electrophysiology. Here in many cases the string galvanometer cannot be replaced by any other instrument.

A number of electrophysiological investigations of the most various kind can be made with the same string. So in the laboratory the same string n°. 18 is now used for investigating the electrocardiogram, cardiac sounds and sounds generally, retinal currents and nerve currents. Yet we will briefly discuss here the conditions which must be fulfilled by a string, chosen from a number of available strings, in order to yield the best results in a certain electrophysiological investigation.

Let us begin with the tracing of the human electrocardiogram. The current may here be derived from both hands. The hands and lower arms are immersed in large porous pots, filled with a solution of Na Cl, placed in glass vessels, containing a solution of $Zn\,SO_4$. In the zinc sulphate solution are amalgamated zinc cylinders, connected by connecting wires with the galvanometer. Under these circumstances the ohmic resistance of the human body varies with different persons from 1000 to 2000 ohms, an amount considerably smaller than the

resistance of a thin, silvered quartz thread. Of the formerly mentioned quartz threads 10, 13 and 14, thread 13 will give the best results in tracing the electrocardiogram, since of this thread the ohmic resistance is smallest. To be sure, the normal sensitiveness of thread 14 is about 1.4 times greater, but the currents, received by this thread from the pulsating heart, will be about twice weaker on account of the greater resistance.

Besides having a smaller ohmic resistance thread 13 has over thread 10 the additional advantage of possessing a smaller air-resistance to its motion. This latter property here plays an important part. For in order to obtain deflections of practicable magnitude, e.g. of 10 to 15 mm., the sensitiveness of the galvanometer must be so adjusted that a potential difference of 10^{-4} volt in the circuit corresponds to 1 mm. ordinate. For obtaining this the quartz thread must be rather feebly stretched, so that the deflections are aperiodic and under these circumstances a diminution of the resistance to the motion of the string will cause a quicker deflection.

Sticking to the condition that a potential difference of 10^{-4} volt shall correspond to 1 mm. ordinate, we trace with string 13 a human electrocardiogram which is almost absolutely accurate.

With string 10 and especially with string 14, however, curves are then recorded which require corrections. Although the amounts of these corrections are small, and do not go beyond a whole millimetre, so that in many cases they may be neglected, it is not superfluous briefly to remember here the cause of these deviations. It is found in the relation between the velocity of the deflection of the galvanometer and the velocity of the oscillations in potential caused by the action of the heart.

The quicker the galvanometer deflects, the more accurate the photogram of the oscillation of potential will be.

The sensitiveness of string 14 must be so adjusted for tracing the human electrocardiogram that about 1 mm. ordinate corresponds to 0.5×10^{-8} amp. Now with this quartz thread the limit of aperiodicity is in a circuit with small external resistance only reached with an about four times greater tension of the string. If by applying a vacuum the resistance to the motion of the string could be diminished so that the limit of aperiodicity were already reached at the first mentioned sensitiveness, when tracing the electrocardiogram the velocity of motion of the string would be considerably increased, so that then also string 14 might reproduce the oscillations of potential with almost absolute accuracy.

We now pass to the discussion of a second example from electro-

physiology, the investigation of the action-currents of a nerve. Here the galvanometer has to fulfill conditions which in many respects differ from those described above. Choosing as our object the nerve of a frog, from which the current must be led to the galvanometer, we shall have to count with a great external resistance, e.g. of 10^5 ohms.

Compared with this the resistance of the galvanometer, even that of thread n°. 14 may be called small. The potential difference caused by the action of the nerve, and available for the current to be measured, is considerably greater than that which is met with in the investigation of the human electrocardiogram, but the duration of a nerve action current is shorter and is measured by only a few thousandths of a second.

These data show us the way in choosing a quartz thread.

In the first place we easily perceive that the differences in the ohmic resistance of the quartz threads can only have an insignificant influence on the intensity of the action current, since the resistance of the nerve itself in the circuit is preponderant. Further, the deflection of the quartz thread must be very quick, hence the tension great; and since an oscillating deflection must be avoided, it will be desirable to adjust the tension so that the motion of the string is brought to the limit of aperiodicity. But even under these circumstances the deflection is not quick enough for accurately reproducing the action current of the nerve. We must therefore apply means that enable us to increase the velocity of deflection without the motion becoming oscillatory. We shall have to try to increase the damping, and can for this purpose apply with good result the "condenser method" formerly described by us. [1])

So we come to requirements here which are opposed to those which we had repeatedly to put in the above described methods. Whereas applying a vacuum had then to be considered an important advantage, now increasing the damping becomes an urgent necessity.

Under these conditions the conception of a normal sensitiveness comes out to its full advantage and it may be briefly stated that of a number of threads of equal ohmic resistance that with the greatest normal sensitiveness is to be preferred. If the external resistance in the circuit is great compared with the resistance of the galvanometer, then of a number of threads with equal normal sensitiveness that with the greatest ohmic resistance will have to be preferred.

For the investigation of the action current of the nerve of a frog,

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[1]) See these "Proceedings" 7, p. 315. 1904.

among the three threads mentioned, n°. 14 will have to be preferred, since the amount of the normal sensitiveness as well as the resistance of this thread exceed those of the two other threads.

Finally we make some remarks as to the manner in which the velocity of deflection may be raised to a maximum. A great velocity is in general obtained at the expense of the sensitiveness. But there are a number of investigations, notably the recording of sounds, [1]) in which the sensitiveness of the string galvanometer may be very considerably diminished. Even when the string, at the risk of breaking, is stretched to its maximum and hence its sensitiveness reduced to a minimum, relatively feeble sounds can still drive the image of the string out of the field of vision.

By strongly stretching string 14 we could impart to it an oscillatory motion of which the period was $T = 1{,}41\ \sigma$. If the oscillations were damped by means of the condenser method, a deflection could be obtained, requiring a time of $0{,}8\ \sigma$ and proportional to the current to be measured with an error of 3 %. [2]) If an accuracy of 0,3 % was desired, one had to be contented with a time of deflection of $2{,}2\ \sigma$. The sensitiveness was here 1 mm deflection for 3×10^{-7} amp.

From the data of the preceding chapter follows that under these conditions the tension of string 14 can be still 3 times increased before its breaking point is reached. Hence if the string is so strongly stretched that it is at the point of breaking, its deflections will become $\sqrt{3}$ times quicker, so that its oscillations will show a period $T = 0{,}815\ \sigma$. In practice we have not raised the tension of string 14 so high, however.

The question how to obtain quicker oscillations was simply solved by using a shorter wire. String 20, which was already discussed above, has a diameter of $1\ \mu$ and is 25 mm long. With a practicable tension that could be applied without risk of breaking, it performed with a sensitiveness of 1 mm deflection for 10^{-5} amp. oscillations of a period of $0{,}31\ \sigma$.

This period corresponds to a tone of 3230 vibrations per second, about g^4 sharp or almost the highest tone of an ordinary piano. We remark that the string can still be shortened and be more strongly stretched, so that a much higher number of vibrations can easily be reached, while it must also be borne in mind that a string with slow deflection can yet very accurately record sound vibrations

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[1]) On the method of recording sounds see these "Proceedings" 6 p. 707, 1904.

[2]) See these" "Proceedings" 7, p. 315, 1904.

of high frequency. So strings 10, 13 and 14 reproduced with feeble tension and slow deflection the sound waves of a tuning fork of 2380 whole vibrations per second. The recorded period was about 24 times shorter than the proper period of the quartz thread. If the same ratio of periods holds for string 20, this latter must be able to reproduce with ease tones of 77000 whole vibrations per second. On a following occasion I hope to return to the recording of sounds. Also a discussion of the practical execution of some of the experiments described above and a description of different designs of the string galvanometer will have to be postponed to a following paper.

Zoology. — "*On a new species of Corallium from Timor.*" By SYDNEY J. HICKSON, Professor of Zoology in the Victoria University of Manchester. (Communicated by Prof. MAX WEBER).

The species of corals included in the family Coralliidae have been arranged by systematists in the four genera, *Corallium*, *Pleurocorallium*, *Hemicorallium* and *Pleurocoralloides*.

The genus *Hemicorallium* of Gray was merged with *Pleurocorallium* by Ridley in 1882, and quite recently KISHINOUYE has called attention to the difficulty there is in maintaining the distinction between *Pleurocorallium* and *Corallium*.

One of the principal characters of *Pleurocorallium* is the presence in the coenenchym of peculiar twinned spicules which Ridley calls "opera-glass" shaped spicules. These "opera-glass" shaped spicules are not supposed to occur in the genus *Corallium*. Whether future investigations will support the view of KISHINOUYE or not is a question which need not be considered here, but the absence of "opera-glass" shaped spicules in the specimen about to be described justifies its position in the genus *Corallium*, that is, to the genus that includes *Corallium nobile* the precious coral of the Mediterranean sea and the seas of the Cape Verde islands and *Corallium japonicum* one of the precious corals of the Japanese seas.

Before proceeding to a description of the new species a few words may be written concerning the geographical distribution of the family. *Corallium nobile* occurs in the Mediterranean sea and off the Cape Verde islands. Some species attributed to the genus *Pleurocorallium* occur off the island Madeira, and quite recently a specimen of *Pseudocorallium johnsoni* has been dredged off the coast of Ireland.

Off the coast of Japan occurs *Corallium japonicum* and several species which would be included on the old system in the genus *Pleurocorallium* but are referred to the genus *Corallium* by KISHINOUYE.

Isolated specimens of Coralliidae were also obtained off Banda in 200 fathoms, the Ki islands 140 fathoms and Prince Edward Island 310 fathoms by the Challenger and there is a doubtful record of a specimen of *Pleurocorallium secundum* from the Sandwich islands. Fisheries of more or less importance have been carried on in the Mediterranean Sea, off the Cape Verde Islands and off the coast of Japan, but there is not, I believe, any historical record of a systematic fishery for precious coral in any other part of the world.

In 1901 the value of the coral obtained off the coast of Japan was over £ 50.000 and it is a fact of considerable interest that a large part of this was exported by the Japanese to Italy.

The coral Fishery of Japan is of very recent growth for in the time of the Daimyos the collection and sale of coral was prohibited, and it was not until the time of the Meji reform 1868 that it began to assume important dimensions.

That the Japanese of old times valued the precious coral is shown in the numerous "Netsukes" and other ornaments which are decorated with it; but the origin of this coral is not definitely known.

On many of the Netsukes the coral is represented in the hands of darkskinned fishermen, "Kurombo"; never in the hands or nets of the Japanese.

Now the art of Japan is quite sufficiently accurate to prove that the Kurombo were not Ainos nor Japanese, nor Malays nor Europeans; but the curly-hair, the broad noses and other features that are consistently shown render it almost certain that the Kurombo were Melanesians or Papuans.

The only regions where such folk live that have hitherto yielded specimens of precious coral are the Banda seas. As already mentioned the Challenger discovered precious coral in deep water off the Banda and Ki islands, but the specimens were "dead" and it was consequently impossible to determine definitely to what species they belong, but they were referred by Ridley to the species *Pleurocorallium secundum*.

In the material that was kindly sent to me by Prof. MAX WEBER from the rich collections of H. M. SIBOGA there were a few small pieces of a beautiful coral which I recognised at once to be a Coralliid. There can be no doubt that it was alive when captured by the dredge and it reached me, not fully expanded, but in a good state of preservation.

The locality of this find was station 280 i. e. at a depth of 1224

metres in the middle of the strait that separates the E. end of the island of Timor from the small island Lette or in other words on the Southern boundary of the Banda Sea.

The axis of this coral is covered with very little or hardly any crust, is apparently as hard as the best Italian coral and is of a good colour, although a little darker than that which is regarded by the jewellers as the best quality.

The discovery of this specimen suggests that the dark skinned "Kurombo" fisherman that supplied the ancient Japanese jewellers with their precious coral, lived some where within the region of Timor. It is of course improbable that they were able to fish in such a great depth as 1224 metres, but as the species of *Corallium* range in depth from 10 fathoms to several hundred fathoms, it is quite possible that they had knowledge of shallow waters off their coast where the coral grew abundantly.

It is not for me to suggest that there is a prospect of a valuable coral fishery in the Banda seas; but now that it is known that living precious coral does occur in deep water in this region of the world it would not be a matter of surprise to scientific men if it were subsequently found at depths sufficiently shallow to be obtained by ordinary fishing boats.

The specimen obtained by the Siboga does not agree exactly with any known Coralliidae in those characters which are used by systematists for the separation of species and it is necessary to find a new name for it, and I should like with Her royal permission to name it *Corallium reginae* in honour of Her Majesty the Queen of Holland whose interest in Zoological Science in general and in the researches of H. M. Siboga in particular has been manifested on more than one occasion.

The specimen agrees with other species of the genus *Corallium* in the absence of the curious "opera glass" shaped spicules and the presence of spicules of the octoradiate type only in the general coenenchym.

It differs from *Corallium* and agrees with many species referred to the genus *Pleurocorallium* in having the branches arranged principally in one plane and the zooids scattered irregularly on one face or surface of this plane.

The autozooids are indicated by well-defined verrucae projecting about 1—5 m.m. from the general surface of the coenenchym. These verrucae are large as compared with other species being about 1—4 m.m. in diameter. The coenenchym is thin, and the axis hard and

either not marked or very faintly marked in some places by longitudinal striations.

The base of the main stem of the specimen is 6 m.m. in diameter and the primary branches are 4—5 m.m. in diameter.

Some further particulars concerning the anatomy of the species will be described with illustrations in a future publication. For the present the diagnosis of the species given above is sufficient. Before concluding this preliminary note I have, with very great regret, to record that on Sept. 22nd a fire broke out in my laboratory and some portions of the specimen were seriously burned and scorched. Fortunately there is still a considerable fragment that appears to be uninjured.

Physics. — "*Properties of the critical line (plaitpoint line) on the side of the components.*" By Prof. VAN DER WAALS.

By CENTNERSZWER and SMITS' observations, by a remark of VAN 'T HOFF and by VAN LAAR's calculations [1]) a discussion has been carried on on the rise of the critical temperature of a substance in consequence of an admixture. In this it has been perfectly overlooked that already more than ten years ago the principal properties of the critical line, and also the properties at the beginning and at the end of this line were discussed and determined by me [2]).

For normal substances, I found by a thermodynamic method, which is a perfectly sure way, for the quantity mentioned the formula (9) (l. c. p. 89)

$$T\left(\frac{dx}{dT}\right)_0 = \frac{\frac{\partial^2 \varepsilon}{\partial v^2}}{\frac{\partial^3 \psi}{\partial x\, \partial v^2} - \frac{1}{MRT}\left(\frac{\partial^2 \psi}{\partial x\, \partial v}\right)^2}.$$

I shall explain further on why I make some reservation for abnormal substances.

And with the aid of the equation of state I derived from (9) formula (11)

$$\frac{d \log T}{dx_0} = \frac{d \log \frac{a}{b}}{dx_0} + \frac{9}{16}\left(\frac{d \log \frac{a}{b^{2/3}}}{dx_0}\right)^2$$

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1) These Proc. p. 144.

2) Verslag Kon. Akad. v. Wet. 25 Mei 1895, p. 20 and 29 Juni 1895, p. 82.

As in this derivation of (11) from (9) the quantity b of the equation of state was supposed constant, (11) must only be considered as an approximation.

If for the present we keep to this form, (11) may also be written:

$$\frac{dT}{Tdx_0} = \frac{dT_x}{T_x\,dx_0} + \frac{9}{16}\left(\frac{dT_x}{T_x\,dx_0} + \frac{1}{3}\frac{db}{b\,dx_0}\right)^2 \quad . \quad . \quad . \quad . \quad (1)$$

And taking into consideration that $b = \frac{T_x}{8 \times 273\, p_x}$ we find finally:

$$\frac{dT}{T\,dx_0} = \frac{dT_x}{T_x\,dx_0} + \left(\frac{dT_x}{T_x\,dx_0} - \frac{1}{4}\frac{dp_x}{p_x\,dx_0}\right)^2 \quad . \quad . \quad . \quad . \quad (2)$$

The quantity T_x occurring in this equation, represents the critical temperature of the unsplit mixture. For this quantity I have already demonstrated in my Théorie Moléculaire that it may get a minimum value for some sorts of mixtures — and the observations of KUENEN, QUINT and others have furnished instances of the existence of such a minimum value. If the admixture should be of such a nature that such a minimum value existed, it would, of course, be perfectly absurd to substitute $T_{x_2} - T_{x_1}$ for $\frac{dT_x}{dx}$. But the existence of such a minimum critical temperature is only to be expected, at any rate only observed, when T_{x_2} and T_{x_1} differ little. When they differ much, $\frac{dT_x}{dx_0}$ can be represented by $T_{x_2} - T_{x_1}$ at least with approximation. As b depends on x linearly at least with some approximation, we may write

$$\frac{1}{b}\frac{db}{dx_0} = \frac{\frac{T_{x_2}}{p_{x_2}} - \frac{T_{x_1}}{p_{x_1}}}{\frac{T_{x_1}}{p_{x_1}}}.$$

Let us with these approximate values compare equation (1) with KEESOM's observations on the mixtures of carbonic acid and oxygen [1]). The critical temperatures of these substances differ sufficiently to enable us to use the approximate values. T_{x_2} (for oxygen) is namely about half of T_{x_1} (that of carbonic acid) — and so we put for $\frac{dT_x}{T_x dx_0}$ the value $\frac{154{,}2 - 304{,}02}{304{,}02} = -0{,}493$, and for $\frac{1}{b}\frac{db}{dx_0}$ the value $\frac{\frac{154{,}2}{50{,}7} - \frac{304{,}02}{72{,}93}}{\frac{304{,}02}{72{,}98}}$ or $-0{,}271$. With these values

[1]) These Proc. VI, p. 616.

we find:

$$\frac{dT}{Tdx_0} = -0{,}493 + \frac{9}{16}(-0{,}493 - 0{,}0903)^2 = -0{,}493 + 0{,}1914 = -0{,}302.$$

The value found by KEESOM for $x = 0{,}1047$ is $\Delta T = -8{,}99$. Supposing this value of x small enough to be substituted for dx_0, we find $\frac{dT}{Tdx_0} = -0{,}284$.

For $x = 0{,}1994$ this value of ΔT found by KEESOM is equal to $-18{,}47$; with these data we should find $\frac{dT}{Tdx_0} = -0{,}304$, so perfectly equal to the value calculated by means of (1). We have here not a molecular increase of the critical temperature, but a decrease, as indeed, was to be expected, because we had to do with the addition of a more volatile component.

Though I derived formula (9), on which formula (11) of 1895 and formula (1) of this communication are founded, in more than one way in my two communications of 1895, I will derive them once more here in order to have an opportunity to discuss somewhat more fully some questions which present themselves in the derivation.

For the plaitpoint line the simple relation:

$$\frac{dp}{dT} = \frac{\left(\frac{\partial^2\eta}{\partial x^2}\right)_{pT}}{\left(\frac{\partial^2 v}{\partial x^2}\right)_{pT}},$$

holds, which, $\left(\frac{\partial^2\eta}{\partial x^2}\right)_{pT}$ not being directly known, may be brought under the following form:

$$T\frac{dp}{dT} = T\left(\frac{dp}{dT}\right)_{vx} + \frac{\left(\frac{\partial^2\varepsilon}{\partial v^2}\right)_{xT}\left(\frac{dv}{dx}\right)^2_{pT} + 2\left(\frac{\partial^2\varepsilon}{\partial x\partial v}\right)_T\left(\frac{dv}{dx}\right)_{pT} + \left(\frac{\partial^2\varepsilon}{\partial x^2}\right)_{vT}}{\left(\frac{d^2v}{dx^2}\right)_{pT}}.$$

The factors of $\left(\frac{dv}{dx}\right)^2_{pT}$ and $\left(\frac{dv}{dx}\right)_{pT}$ and also $\left(\frac{\partial^2\varepsilon}{dv^2}\right)_{vT}$ being finite quantities, and on the other hand $\left(\frac{dv}{dx}\right)_{pT}$ being infinitely great, when the plaitpoint lies at $x = 0$, we may write for this case:

$$T\frac{dp}{dT} = T\left(\frac{\partial p}{dT}\right)_{vx} + \left(\frac{\partial^2\varepsilon}{\partial v^2}\right)_{xT}\frac{\left(\frac{dv}{dx}\right)^2_{pT}}{\left(\frac{d^2v}{dx^2}\right)_{pT}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we put $\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial v^2}-\left(\frac{\partial^2\psi}{\partial x\partial v}\right)^2=f$, then $f=0$, because the plaitpoint is a point of the spinodal line. In the same way:

$$\frac{\partial f}{\partial v}\left(\frac{dv}{dx}\right)_{pT}+\frac{\partial f}{dx}=0$$

because it concerns a plaitpoint.

If we multiply the numerator and the denominator of the fraction occurring in (3) by $\left(\frac{\partial^2\psi}{\partial v^2}\right)^2$, we get:

$$T\frac{dv}{dT}=T\left(\frac{dp}{dT}\right)_{vx}+\left(\frac{\partial^2\varepsilon}{\partial v^2}\right)_{xT}\frac{\left(\frac{\partial^2\psi}{\partial x\partial v}\right)^2}{\left(\frac{\partial^2\psi}{dv^2}\right)^2\left(\frac{d^2v}{dx^2}\right)_{pT}}.$$

The value of $\left(\frac{d^2v}{dx^2}\right)_{pT}\left(\frac{\partial^2\psi}{\partial v^2}\right)^2$ we derive from:

$$\left(\frac{dv}{dx}\right)_{pT}=-\frac{\left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)}{\frac{\partial^2\psi}{\partial v^2}}$$

and find then:

$$-\left(\frac{\partial^2\psi}{\partial v^2}\right)^2\left(\frac{d^2v}{dx^2}\right)_{pT}=\frac{\partial^2\psi}{\partial v^2}\frac{\partial^3\psi}{\partial x^2\partial v}-2\frac{\partial^2\psi}{\partial x\,\partial v}\frac{\partial^3\psi}{\partial x\,\partial v^2}+\frac{\frac{\partial^3\psi}{\partial v^3}}{\frac{\partial^2\psi}{\partial v^2}}\left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)^2$$

As for the critical point of a component both $\frac{\partial^2\psi}{\partial v^2}$ and $\frac{\partial^3\psi}{\partial v^3}$ is equal to 0, the last equation becomes:

$$-\left(\frac{\partial^2\psi}{\partial v^2}\right)^2\left(\frac{d^2v}{dx^2}\right)_{pT}=\left(\frac{\partial^2\psi}{\partial x\partial v}\right)^2 lim\frac{\frac{\partial^3\psi}{\partial v^3}}{\frac{\partial^2\psi}{\partial v^2}}-2\frac{\partial^2\psi}{\partial x\partial v}\frac{\partial^3\psi}{\partial x\,\partial v^2}.$$

The limiting value of $\frac{\frac{\partial^3\psi}{\partial v^3}}{\frac{\partial^2\psi}{\partial v^2}}$ can be found from the equation which expresses that the critical point of the component is a plaitpoint, viz.:

$$\frac{\partial f}{\partial v}\frac{\partial^2\psi}{\partial x\partial v}=\frac{\partial f}{\partial x}\frac{\partial^2\psi}{\partial v^2}\ ^{1}).$$

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[1]) In a derivation of the discussed formula in my communications of 1895 I put $\frac{\partial f}{\partial v}=0$. It would have been more accurate, if I had put this quantity infinitely small compared to $\frac{\partial f}{\partial x}$.

Now

$$\frac{\partial f}{\partial v} = \frac{\partial^3 \psi}{\partial x^2 \partial v} \frac{\partial^2 \psi}{\partial v^2} + \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^3 \psi}{\partial v^3} - 2 \frac{\partial^2 \psi}{\partial x \partial v} \frac{\partial^3 \psi}{\partial x \partial v^2}$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial^3 \psi}{\partial x^3} \frac{\partial^2 \psi}{\partial v^2} + \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^3 \psi}{\partial x \partial v^2} - 2 \frac{\partial^2 \psi}{\partial x \partial v} \frac{\partial^3 \psi}{\partial x^2 \partial v}$$

or taking into consideration $\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial v^2} = \left(\frac{\partial^2 \psi}{\partial x \partial v}\right)^2$ and $\frac{\partial^2 \psi}{\partial v^2} = 0$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial^2 \psi}{\partial x \partial v}\right)^2 lim \frac{\frac{\partial^3 \psi}{\partial v^3}}{\frac{\partial^2 \psi}{\partial v^2}} - 2 \frac{\partial^2 \psi}{\partial x \partial v} \frac{\partial^3 \psi}{\partial x \partial v^2}$$

and

$$\frac{\partial^2 \psi}{\partial v^2} \frac{\partial f}{\partial x} = \left(\frac{\partial^2 \psi}{\partial x \partial v}\right)^4 lim \frac{\frac{\partial^3 \psi}{\partial x^3}}{\left(\frac{\partial^2 \psi}{\partial x^2}\right)^2} + \left(\frac{\partial^2 \psi}{\partial x \partial v}\right)^2 \frac{\partial^3 \psi}{\partial x \partial v^2}.$$

By equating $\frac{\partial f}{\partial v} \frac{\partial^2 \psi}{\partial x \partial v}$ and $\frac{\partial f}{\partial x} \frac{\partial^2 \psi}{\partial v^2}$ we find:

$$\frac{\partial^2 \psi}{\partial x \partial v} \; lim. \; \frac{\frac{\partial^3 \psi}{\partial v^3}}{\frac{\partial^2 \psi}{\partial v^2}} - 2 \frac{\partial^3 \psi}{\partial x \partial v^2} = \left(\frac{\partial^2 \psi}{\partial x \partial v}\right)^2 \; lim. \; \frac{\frac{\partial^3 \psi}{\partial x^3}}{\left(\frac{\partial^2 \psi}{\partial x^2}\right)^2} + \frac{\partial^3 \psi}{\partial x \partial v^2}.$$

For normal substances the limiting value of $\frac{\frac{\partial^3 \psi}{\partial x^3}}{\left(\frac{\partial^2 \psi}{\partial x^2}\right)^2}$ is known.

From $\psi = MRT\{(1-x)\, l\,(1-x) + xlx\} - \int p dv$ follows:

$$\left(\frac{\partial \psi}{\partial x}\right)_v = MRTl \frac{x}{1-x} - \int \frac{\partial p}{\partial x} dv$$

$$\left(\frac{\partial^2 \psi}{\partial x^2}\right) = MRT \frac{1}{x(1-x)} - \int \frac{\partial^2 p}{\partial x^2} dv$$

$$\left(\frac{\partial^3 \psi}{\partial x^3}\right) = - \frac{MRT(1-2x)}{x^2(1-x)^2} - \int \frac{\partial^3 p}{\partial x^3} dv$$

for $x = 0$ we get $\frac{\frac{\partial^3 \psi}{\partial x^3}}{\left(\frac{\partial^2 \psi}{\partial x^2}\right)^2} = - \frac{1}{MRT}$.

For abnormal substances this quantity would probably be found of the same value, but this would require a closer investigation, into which I shall not enter here. For this reason I have made a reservation for abnormal substances above.

For the value of $\left(\frac{dp}{dT}\right)_{x_0}$ of the plaitpoint curve we get now the equation:

$$T\left(\frac{dp}{dT}\right)_{x_0} = T\left(\frac{\partial p}{\partial T}\right)_{vx} + \frac{\partial^2\varepsilon}{\partial v^2}\frac{\left(\frac{\partial^2\psi}{\partial x\partial v}\right)}{\left(\frac{\partial^2\psi}{\partial x\partial v}\right)^2\frac{1}{MRT} - \frac{\partial^3\psi}{\partial x\partial v^2}} \quad . \quad . \quad (4)$$

The critical point of the component is an homogeneous phase, in the same way the plaitpoint is a new homogeneous phase. But the quantities T_x, v_x and $x = 0$ increase by dT_x, dv_x and dx_0. Hence:

$$dp = \left(\frac{\partial p}{\partial T}\right)_{vx} dT_x + \left(\frac{\partial p}{\partial x}\right)_{vT} dx_0 + \left(\frac{\partial p}{\partial v}\right)_{xT} dv_x$$

and $\left(\frac{\partial p}{\partial v}\right)_{xT}$ being 0, also

$$T\left(\frac{dp}{dT}\right)_{x_0} = T\left(\frac{\partial p}{\partial T}\right)_{vx} + \left(\frac{\partial p}{\partial x}\right)_{vT}\frac{Tdx_0}{dT}$$

comparing with (4) we find the value sought of

$$\frac{dT}{Tdx_0} = \frac{\frac{\partial^2 p}{\partial x\partial v} + \frac{1}{MRT}\left(\frac{\partial p}{\partial x}\right)_{vT}}{-\frac{\partial^2\varepsilon}{\partial v^2}}$$

According to the equation of state, supposing b constant, we get: $\varepsilon = -\frac{a}{v}$, $\left(\frac{\partial\varepsilon}{\partial v}\right) = \frac{a}{v^2}$ and $\frac{\partial^2\varepsilon}{\partial v^2} = -\frac{2a}{v^3}$. The value of $\left(\frac{\partial p}{\partial x}\right)_{vT}$ is equal to $\left\{\frac{MRT}{(v-b)^2}\frac{db}{dx} - \frac{da}{dx}\frac{1}{v^2}\right\}$ and of $\left(\frac{\partial^2 p}{\partial x\partial v}\right)$ equal to $-2\left\{\frac{MRT}{(v-b)^3}\frac{db}{dx} - \frac{da}{dx}\frac{1}{v^3}\right\}$, and so

$$\frac{dT}{Tdx_0} = \frac{2\left\{\frac{da}{dx}\frac{1}{v^3} - \frac{MRT}{(v-b)^3}\frac{db}{dx}\right\} + \frac{1}{MRT}\left\{\frac{da}{dx}\frac{1}{v^2} - \frac{MRT}{(v-b)^2}\frac{db}{dx}\right\}^2}{\frac{2a}{v^3}}$$

or

$$\frac{dT}{Tdx_0} = \left\{\frac{da}{adx} - \frac{MRT}{a}\frac{v^3}{(v-b)^3}\frac{db}{dx}\right\} + \frac{a}{2vMRT}\left\{\frac{1}{a}\frac{da}{dx} - \frac{MRT}{a}\frac{v^2}{(v-b)^2}\frac{db}{dx}\right\}^2.$$

If we put, as is found with constant value of b, $v = 3b$ and $MRT = \frac{8}{27}\frac{a}{b}$, we find the value given above:

$$\frac{d\log T}{dx_0} = \frac{d\log \frac{a}{b}}{dx} + \frac{9}{16}\left(\frac{d\log \frac{a}{b^2/_3}}{dx_0}\right)^2$$

In what precedes the relation between the variation of T, and that of x has been discussed for the beginning of the plaitpoint line. Let us now proceed to the discussion of the relation between the variation of p and that of T.

From the equation of $\frac{dp}{dT}$ given above, we derive:

$$\left(\frac{T}{p}\frac{dp}{dT}\right)_{x_0} = \frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_{vx} + \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}\frac{T dx_0}{dT}.$$

Now in the critical point of a component $\left(\frac{\partial p}{\partial T}\right)_v$ is equal to $\frac{dp}{dT}$ for the saturated vapour tension. And for numerous substances $\frac{T}{p}\frac{dp}{dT}$ for the saturated vapour tension is about 7 in the critical point. $\frac{T dx_0}{dT}$ now being known, we want still the knowledge of $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$ for the calculation of $\left(\frac{T}{p}\frac{dp}{dT}\right)_{x_0}$ for the plaitpoint line.

We can calculate $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$ by means of the equation of state.

If we put again b constant, we find the value indicated above:

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = \frac{1}{p}\left\{\frac{MRT}{(v-b)^2}\frac{db}{dx} - \frac{da}{dx}\frac{1}{v^2}\right\}$$

or

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = \frac{\frac{a}{v^2}}{p}\left\{\frac{8}{27}\frac{1}{b}\frac{db}{dx}\frac{v^3}{(v-b)^2} - \frac{1}{a}\frac{da}{dx}\right\}.$$

With $v = 3b$ and $p = \frac{1}{27}\frac{a}{b^2}$ we should find for carbonic acid and oxygen

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = 3 \times (0{,}493 + 0{,}090\beta)\ ^1)$$

or

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1) See page 273.

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = 1{,}75.$$

According to KEESOM's observations the value of $\left(\frac{Tdp}{pdT}\right)$ for the beginning of the plaitpoint line is equal to $-6{,}3$ for $x = 0{,}1047$, and equal to $-6{,}08$ for $x = 0{,}1995$. From this we calculate, with $\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_{vx} = 6{,}7$ (the value found for carbonic acid) $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right) = 3{,}921$ and $3{,}824$ — so more than double the value which follows from the equation of state, when we put there b independent of the volume.

The values given by KEESOM for pressure and temperature of the critical tangent point, and of the critical point of the unsplit mixture, furnish a means to test the reliability of the value of $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$, as it has been calculated from his observations.

For the mixture $x = 0{,}1047$, Δp amounted to 9,9 for the critical tangent pressure and ΔT to $-7{,}69$ for the critical tangent temperature.

If we again write for this homogeneous phase:

$$dp = \left(\frac{\partial p}{\partial T}\right)_{vx} dT + \left(\frac{\partial p}{\partial x}\right)_{vT} dx$$

or

$$\frac{1}{p}\frac{dp}{dx} = \left(\frac{T}{p}\frac{\partial p}{\partial T}\right)_{vx}\frac{dT}{Tdx} + \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT},$$

we find:

$$\frac{9{,}9}{72{,}93 \times 0{,}1047} = 6{,}7\,\frac{-7{,}69}{304{,}02 \times 0{,}1047} + \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$$

or

$$1{,}297 + 1{,}62 = \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = 2{,}917.$$

And from the observations for $x = 0{,}1994$

$$\frac{16{,}72}{72{,}93 \times 0{,}1944} = 6{,}7\,\frac{-14{,}71}{304{,}02 \times 0{,}1994} + \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$$

or

$$1{,}15 + 1{,}635 = \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = 2{,}785.$$

For the homogeneous phase of the critical circumstances of the unsplit mixture which cannot be realized, KEESOM found $\Delta p = -5{,}23$ and $\Delta T = -18{,}34$ by the application of the law of the corresponding states. From these data we find:

$$\frac{-5{,}23}{72{,}93 \times 0{,}1047} = 6{,}7\,\frac{-18{,}34}{304{,}02 \times 0{,}1047} + \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$$

or

$$-0{,}685 + 3{,}86 = \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = 3{,}18.$$

The fact that Δx, ΔT and Δp cannot be considered as differentials will undoubtedly contribute to the circumstance that this quantity shows such different values if calculated from KEESOM's observations.

But though the calculated values for $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$ are not the same, it appears sufficiently that the value of this quantity lies in the neighbourhood of 3, and probably above it. That the equation of state gives a so much lower value if we put b constant, must be attributed to the fact that the influence of this erroneously introduced simplification is great here, whereas this simplification caused hardly any error in the calculation of $\frac{T\,dx_0}{dT}$. The value of $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$ we found equal to:

$$-\frac{a}{v^2 p_x}\left\{\frac{1}{a}\frac{da}{dx} - \frac{8}{27}\frac{v^3}{(v-b)^3}\,\frac{1}{b}\frac{db}{dx}\right\}.$$

With $v_x = 3\,b$ we find the value 3 for $\frac{a}{v^2 p_x}$, while the second factor becomes equal to $\frac{dl\frac{a}{b}}{dx} + \frac{1}{3}\frac{dlb}{dx}$. But it is sufficiently known that the critical volume is much smaller than $3\,b$, and that the variability of b accounts for it. The same cause to which it is due that at the critical volume $\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_v$ is found equal to $1+6$, instead of $1+3$, causes us to find $\frac{a}{v^2 p_x}$ equal to 6 instead of to 3. Let us briefly prove this.

$$T\left(\frac{\partial p}{\partial T}\right)_v = \frac{MRT}{v-b} = p + \frac{a}{v^2}$$

$$\frac{T}{p}\left(\frac{\partial p}{\partial T}\right) = 1 + \frac{a}{pv^2}$$

In the critical circumstances the value of the first member is about 7 or $\frac{a}{p_x v_x^2} = 6$. If we use this value, we find for $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$ double the previous value, i. e. 3,5. The second factor of $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT}$

will now have to suffer some modification too, and as I shall show in another communication, be equal to $\frac{1}{a}\frac{da}{dx} - \frac{5}{6}\frac{1}{b}\frac{db}{dx} = \frac{dl\frac{a}{b}}{dx} + \frac{1}{6}\frac{1}{b}\frac{db}{dx}$, but the difference is slight — and this factor and others of a similar form occurring in the value of $\frac{dT}{Tdx}$, the value of $\frac{dT}{Tdx}$, calculated on the supposition of b variable, may be considered as sufficiently accurate.

We must therefore not expect to find a perfectly complete discussion of the problem in what precedes. If this was wanted, a closer investigation would be required for the determination of $\left(\frac{\partial p}{\partial x}\right)_v$ and $\frac{\partial^2 p}{\partial x\,\partial v}$, if we take b dependent, not only on x, but also on v —, and hence put:

$$b = b_\infty \left\{1 + \alpha\left(\frac{b_\infty}{v}\right) + \beta\left(\frac{b_\infty}{v}\right)^2 \text{etc.}\right\},$$

while $b_\infty = (b_1)_\infty\ (1-x) + (b_2)_\infty\, x$ is put. But in the following communication I shall show that in this particular case, the component being in critical circumstances, we can determine the value of these quantities without entering into a closer investigation.

Physics. "*The properties of the sections of the surface of saturation of a binary mixture on the side of the components.*" By Prof. VAN DER WAALS.

I have brought the differential equation of the p,x,T-surface of a binary mixture into the following form:

$$v_{21}\,dp = (x_2 - x_1)\left(\frac{\partial^2 \zeta}{\partial x_1^2}\right)_{pT} dx_1 + \frac{w_{21}}{T}\,dT.$$

In this equation $\left(\frac{\partial^2 \zeta}{\partial x_1^2}\right)_{pT}$ is equal to $\left(\frac{\partial^2 \psi}{\partial x_1^2}\right)_{vT} - \frac{\left(\frac{\partial^2 \psi}{\partial x_1 \partial v}\right)^2}{\frac{\partial^2 \psi}{\partial v^2}}$.

For x_1 infinitely small $v_{21} = v_2 - v_1$, $\left(\frac{\partial^2 \zeta}{\partial x_1^2}\right)_{pT} = \left(\frac{\partial^2 \psi}{\partial x_1^2}\right)_{vT} = \frac{MRT}{x_1(1-x_1)}$ and for w_{21} we may substitute the molecular heat of evaporation of the component, which we shall denote by Mr. The above equation is then simplified to

$$(v_2-v_1)\ dp = \frac{MRT}{x_1}(x_2-x_1)\,dx_1 + \frac{Mr}{T}\,dT.$$

The properties of the initial direction of the sections normal to the T-axis, normal to the p-axis and normal to the x-axis are given by this equation and they are known when the value of $\frac{x_2-x_1}{x_1}$ is known.

If x_1 and v_1 represent the value of x and of the molecular volume of the liquid phase, and in the same way x_2 and v_2 these quantities for the vapour phase, then the equation:

$$(v_2-v_1)\,dp = MRT\,\frac{x_2-x_1}{x_2}\,dx_2 + \frac{Mr}{T}\,dT$$

holds for the vapour phase.

As the difference of the specific volumes of liquid and vapour of a component is generally represented by u, $v_2-v_1 = Mu$, and this equation might also be written

$$udp = RT\,\frac{x_2-x_1}{x_1}\,dx_1 + \frac{r}{T}\,dT.$$

For the section normal to the x-axis, so for the component itself, we find the well-known equation of CLAPEYRON:

$$\frac{r}{u} = T\frac{dp}{dT}.$$

For this section it is not required to know $\frac{x_2}{x_1}$, but for the other sections it is indispensable.

This relation is found by means of the property which says, that $\left(\frac{\partial\psi}{\partial x}\right)_{vT}$ must have the same value for liquid and vapour phase.

From:

$$\psi = MRT\,\{(1-x)\,l\,(1-x) + x l x\} - \int^{v} p\,dv + F(T)$$

we find:

$$\left(\frac{\partial\psi}{\partial x}\right)_{vT} = MRT\,l\,\frac{x}{1-x} - \int\frac{\partial p}{\partial x}\,dv,$$

and equating this value for the two phases we get:

$$MRT\,l\,\frac{x_1}{1-x_1} - \int^{v_1}\left(\frac{\partial p}{\partial x}\right)_{vT}dv = MRT\,l\,\frac{x_2}{1-x_2} - \int^{v_2}\left(\frac{\partial p}{\partial x}\right)_{vT}dv$$

or

$$MRT\,l\,\frac{x_1}{x_2}\,\frac{1-x_2}{1-x_1} = \int_{v_2}^{v_1}\left(\frac{\partial p}{\partial x}\right)_{vT}dv,$$

and so for x_1 and x_2 infinitely small:

$$MRT\, l \frac{x_2}{x_1} = \int_{v_1}^{v_2} \left(\frac{\partial p}{\partial x}\right)_{vT} dv.$$

If we represent the mean value of $\left(\frac{\partial p}{\partial x}\right)_{vT}$ between the values v_1 and v_2 by $\overline{\left(\frac{\partial p}{\partial x}\right)_{vT}}$, we may also write:

$$MRT\, l \frac{x_2}{x_1} = (v_2 - v_1) \overline{\left(\frac{\partial p}{\partial x}\right)_{vT}}$$

This mean value can also be represented under another form by the following consideration. According to Maxwell's rule

$$p_c (v_2 - v_1) = \int_{v_1}^{v_2} p dv,$$

when p_c denotes the tension of the saturated vapour of the component, from which follows:

$$\frac{\partial p_c}{\partial x}(v_2 - v_1) + p_c \frac{d(v_2 - v_1)}{dx} = \int_{v_1}^{v_2} \left(\frac{\partial p}{\partial x}\right)_{vT} dv + p_c \frac{d(v_2 - v_1)}{dx}$$

or

$$\frac{\partial p_c}{\partial x}(v_2 - v_1) = \int_{v_1}^{v_2} \left(\frac{\partial p}{\partial x}\right)_{vT} dv = (v_2 - v_1) \overline{\left(\frac{\partial p}{\partial x}\right)_{vT}}.$$

The quantity $\frac{\partial p_c}{\partial x}$ represents the molecular increase of the tension of the saturated vapour for the unsplit mixture, assuming for p_c the approximate value

$$l p_c = l p_x - f \frac{T_x - T}{T}$$

$\frac{\partial p_c}{\partial x}$ is found from:

$$\frac{1}{p_c} \frac{\partial p_c}{\partial x} = \frac{1}{p_x} \frac{\partial p_x}{\partial x} - \frac{f}{T} \frac{\partial T_x}{\partial x}$$

For the present let us continue to write:

$$MRT\, l \frac{x_2}{x_1} = (v_2 - v_1) \frac{\partial p_c}{\partial x}$$

or

$$\frac{x_2}{x_1} = e^{\frac{(v_2 - v_1) \frac{\partial p_c}{\partial x}}{MRT}}$$

Let us now first consider the initial direction of the sections normal to the T-axis. It can now be found from:

$$log\left\{1+\frac{(v_2-v_1)}{MRT}\frac{dp}{dx_1}\right\}=\frac{v_2-v_1}{MRT}\frac{\partial p_c}{\partial x}$$ for the liquid branch and

$$log\left\{1-\frac{v_2-v_1}{MRT}\frac{dp}{dx_2}\right\}=-\frac{v_2-v_1}{MRT}\frac{\partial p_c}{\partial x}$$ for the vapour branch.

For very low temperatures we may put $\frac{p(v_2-v_1)}{MRT}$ about equal to unity, and so:

$$log\left\{1+\frac{dlp}{dx_1}\right\}=\frac{\partial lp_c}{\partial x}=\left[\frac{\partial lp_x}{\partial x}-\frac{f}{T}\frac{\partial T_x}{\partial x}\right]$$

and

$$log\left\{1-\frac{dlp}{dx_2}\right\}=-\frac{\partial lp_c}{\partial x}=-\left[\frac{\partial lp_x}{\partial x}-\frac{f}{T}\frac{\partial T_x}{\partial x}\right]$$

For the case that $\frac{\partial p_c}{\partial x}=0$, $\frac{dp}{dx_1}$ and $\frac{dp}{dx_2}$ is also equal to 0, and it may therefore happen, that the two branches of the p,x-line touch in the beginning, and that both have an horizontal tangent.

As condition for this circumstance we have:

$$\frac{f}{T}\frac{dT_x}{dx}=\frac{dlp_x}{dx}$$

which may also be written:

$$f\frac{T_x}{T}\left(\frac{1}{a}\frac{da}{dx}-\frac{1}{b}\frac{db}{dx}\right)=\left(\frac{1}{a}\frac{da}{dx}-\frac{2}{b}\frac{db}{dx}\right)$$

or

$$f\frac{T_x}{T}\frac{dlT_x}{dx}=\frac{dlT_x}{dx}-\frac{dlb}{dx}$$

or

$$\left(f\frac{T_x}{T}-1\right)\frac{dlT_x}{dx}=-\frac{dlb}{dx}$$

E.g. for $T=\frac{1}{2}T_x$

$$\frac{dlT_x}{dx}+\frac{1}{13}\frac{dlb}{dx}=0.$$

For higher temperatures $\frac{p(v_2-v_1)}{MRT}$ is smaller than unity and for the critical temperature of the component this quantity is even equal to 0. In this case we may write $\frac{v_2-v_1}{MRT}\frac{dp}{dx_1}$ for $log\left(1+\frac{v_2-v_1}{MRT}\frac{dp}{dx_1}\right)$ and we find

$$\frac{dp}{dx_1}=\frac{\partial p_c}{\partial x},$$

and in the same way

$$\frac{dp}{dx_2} = \frac{\partial p_c}{\partial x}.$$

The first conclusion we draw from this is, that at the critical temperature the liquid branch and the vapour branch have always the same tangent, and therefore touch. The initial direction is given by the quantity $\frac{\partial p_c}{\partial x}$ or by $\overline{\left(\frac{\partial p}{\partial x}\right)_{cT}}$. But as at the critical temperature $v_2 = v_1$, the mean value of $\left(\frac{\partial p}{\partial x}\right)_{cT}$ is equal to the value which $\left(\frac{\partial p}{\partial x}\right)_{vT}$ has at that volume equal for vapour and liquid. We have therefore at the critical temperature:

$$\left(\frac{dp}{dx_1}\right)_T = \left(\frac{dp}{dx_2}\right)_T = \left(\frac{\partial p}{\partial x}\right)_{vT}$$

or

$$\frac{1}{p}\left(\frac{dp}{dx_1}\right)_T = \frac{1}{p}\left(\frac{dp}{dx_2}\right)_T = \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{cT} = \frac{\partial lp_c}{\partial x} = -(f-1)\frac{dlT_x}{dx} + \frac{dlb}{dx}$$

The second conclusion we draw is that at the critical temperature

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = -6\left\{\frac{dlT_x}{dx} + \frac{1}{6}\frac{dlb}{dx}\right\}.$$

which has been put in the preceding communication, but has not been proved there.

That at the critical point $\left(\frac{dp}{dx_1}\right)_T$ and $\left(\frac{dp}{dx_2}\right)_T$ is equal to $\left(\frac{\partial p}{\partial x}\right)_{cT}$ we might have immediately concluded, without following the elaborate way by which we have now arrived at this conclusion. In the same way that at the critical point $\left(\frac{dp}{dT}\right)_x = \left(\frac{\partial p}{\partial T}\right)_{vx}$.

Let us first consider a simple substance. If we pass from one homogeneous phase to another, at which v is increased by dv, and T by dT, then

$$dp = \left(\frac{\partial p}{\partial v}\right)_T dv + \left(\frac{\partial p}{\partial T}\right)_v dT.$$

If $\left(\frac{\partial p}{\partial v}\right)_T = 0$, as is the case at the critical point, then:

$$dp = \left(\frac{\partial p}{\partial T}\right)_v dT,$$

and so every $\frac{dp}{dT} = \left(\frac{\partial p}{\partial T}\right)_v$, also with such a change at which the

volume changes, as is the case with saturated vapour. From this follows the well known property that at the critical point $\frac{T}{p}\left(\frac{dp}{dT}\right)$ for saturated vapour is equal to $\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_r$.

If with a binary mixture we pass from one homogeneous phase to another, at which v is increased by dv, T by dT and x by dx, then:

$$dp = \left(\frac{\partial p}{\partial v}\right)_{xT} dv + \left(\frac{\partial p}{\partial T}\right)_{vx} dT + \left(\frac{\partial p}{\partial x}\right)_{vT} dx.$$

If $\left(\frac{\partial p}{\partial v}\right)_{xT} = 0$, as is the case at the critical point of the component, then:

$$dp = \left(\frac{\partial p}{\partial T}\right)_{vx} dT + \left(\frac{\partial p}{\partial x}\right)_{vT} dx$$

also for such variations in which the volume changes.

The differential equation of the surface of saturation:

$$v_{21}\, dp = \frac{w_{21}\, dT}{T} + (x_2 - x_1)\left(\frac{\partial_2 \zeta}{\partial x_1^{\,2}}\right)_{pT} dx_1$$

holds for the transition of an homogeneous liquid phase to a subsequent one and in the same way:

$$v_{12}\, dp = \frac{w_{12}\, dT}{T} + (x_1 - x_2)\left(\frac{\partial^2 \zeta}{\partial x_2^{\,2}}\right)_{pT} dx_2$$

for the transition of an homogeneous vapour phase to a subsequent one.

If the first liquid phase and the first vapour phase is the critical phase of the component, the three last equations must be identical, and so $\frac{w_{21}}{Tv_{21}} = \frac{w_{12}}{Tv_{12}} = \left(\frac{\partial p}{\partial T}\right)_{rx}$, or $\frac{r}{Tu} = \frac{-r}{-Tu} = \left(\frac{\partial p}{\partial T}\right)_{rx}$.

In the same way $\frac{x_2 - x_1}{v_{21}}\left(\frac{\partial^2 \zeta}{\partial x_1^{\,2}}\right)_{pT} = \frac{x_1 - x_2}{v_{12}}\left(\frac{\partial^2 \zeta}{\partial x_2^{\,2}}\right)_{pT} = \left(\frac{\partial p}{\partial x}\right)_{vT}$, as has been proved above as holding for the critical point of the component.

From the general equation:

$$\frac{x_2}{x_1} = e^{\frac{v_2 - v_1}{MRT}\overline{\left(\frac{\partial p}{\partial x}\right)_{vT}}}$$

follows, when $v_2 - v_1$ is infinitely small,

$$\frac{x_2 - x_1}{x_1} = \frac{v_2 - v_1}{MRT}\overline{\left(\frac{\partial p}{\partial x}\right)_{vT}},$$

and

$$\left(\frac{x_2-x_1}{x_1}\frac{MRT}{v_2-v_1}\right)_x=\left(\frac{\partial p}{\partial x}\right)_{vT}.$$ [1])

If at lower temperatures the initial direction of the p,x-line is traced for the liquid phase, and also that for the vapour phase, then these directions are usually different. Between these two directions lies the direction for the line which denotes the course of the quantity p_c. If this last line is an ascending one, this is also the case for the two others, and reversely. If the admixture is called more volatile

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[1]) Though it falls outside the scope of our subject, which only treats of properties on the side of the surface of saturation, I will make a single remark on the mixtures for which liquid and vapour have the same concentration, because these mixtures have many properties which the components also possess. Also for these mixtures the equation: $MRTl\frac{x_2}{x_1}=\int_{v_1}^{v_2}\left(\frac{\partial p}{\partial x}\right)_{vT}dv$, or $MRTl\frac{x_2}{x_1}\overline{\left(\frac{\partial p}{\partial x}\right)_{vT}}$ holds. For these cases $\frac{x_2}{x_1}=1$ and so $\overline{\left(\frac{\partial p}{\partial x}\right)_{vT}}=0$. So for a mixture for which this equation would hold at the critical circumstances, $\left(\frac{\partial p}{\partial x}\right)_{vT}$ itself would be equal to 0. As also $\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial v^2}=\left(\frac{\partial p}{\partial x}\right)_{vT}^2$ must be equal to 0, also $\left(\frac{\partial p}{\partial v}\right)_{vT}=0$, and from:

$$dp=\frac{\partial p}{\partial v_x}_T dv+\left(\frac{\partial p}{\partial x}\right)_{vT}dx+\left(\frac{\partial p}{\partial T}\right)_{vx}dT$$

follows:

$$\frac{dp}{dT}=\left(\frac{\partial p}{\partial T}\right)_{vx}.$$

Already in 1895 I made the remark, which follows from this, viz. that for the point of the plaitpoint curve, at which the line which is sometimes called the line of Konowalow, meets the plaitpoint line, contact must take place and that just as for a simple substance $\frac{T}{p}\frac{dp}{dT}=$ is about 7.

Now I will add that from $\left(\frac{\partial p}{\partial x}\right)_{vT}=0$ follows in the same way, as has been derived above, that:

$$\frac{d\,log\,T_x}{dx}+\frac{1}{6}\frac{d\,log\,b}{dx}=0 \text{ is,}$$

and not

$$\frac{d\,log\,T_x}{dx}+\frac{1}{3}\frac{d\,log\,b}{dx}=0,$$

as would follow when b is put constant. Already Quint pointed out that the last equation was not satisfied in his observations. According to an oral communication the equation given here would be in much better harmony with his observations.

than the component, when it causes the quantity p_c to decrease, then the general rule holds, that both $\frac{dp}{dx_1}$ and $\frac{dp}{dx_2}$ are positive when the admixture is more volatile than the pure substance and reversely.

In general these three directions approach each other at higher temperature and at the critical temperature they coincide. An exception to the rule that with rising temperature the lines approach, must be allowed for the case that for certain value of T the quantity $\frac{\partial p_c}{\partial x} = 0$. In this case the three directions mentioned coincide at that value of T; as they must again coincide at $T = T_x$, and as the quantity $\frac{\partial p_c}{\partial x}$ varies with T, they will first diverge up to a certain maximum amount, and finally approach each other again.

The rule about the approaching of the lines might also be represented in the following manner, which would render my meaning more precisely. If we write $\frac{\partial p}{\partial x_1}$ under the following form, which follows directly from the above:

$$\frac{\frac{dp}{dx_1}}{\frac{dp_c}{dx}} = \frac{e^{\frac{p_c(v_2-v_1)}{MRT}\frac{dlp_c}{dx}} - 1}{\frac{p_c(v_2-v_1)}{MRT}\frac{dlp_c}{dx}},$$

or putting $\frac{p_c(v_2-v_1)}{MRT}\frac{dlp_c}{dx} = k$ and $\frac{\frac{dp}{dx_1}}{\frac{dp_c}{dx}} = U$

$$U = \frac{e^k - 1}{k}$$

from which follows:

$$\frac{dU}{dT} = \frac{1}{k}\frac{dk}{dT}\left\{e^k - \frac{e^k-1}{k}\right\}.$$

The factor $\left(e^k - \frac{e^k-1}{k}\right)$ is always positive for k positive, and always negative for k negative, and is only equal to 0 for $k = 0$; and k equal to 0 occurs only at the critical temperature and when $\frac{\partial p_c}{\partial x} = 0$. For all other values of k can $\frac{dU}{dT}$ only be equal to 0, when $\frac{dk}{dT}$ is equal to 0. When this quantity is equal to 0, there is a maximum

or a minimum value for $\frac{\frac{dp}{dx_1}}{\frac{dp_c}{dx}}$ and the variation of this quantity can reverse its sign with rising temperature. Reversely when $\frac{dk}{dT}$ cannot be equal to 0, reversal of sign cannot take place in the variation of this quantity.

From $k = \frac{p_c(v_2 - v_1)}{MRT} \frac{dlp_c}{dx}$ follows as condition of $\frac{dk}{dT} = 0$.

$$\frac{dlp_c}{dx} \frac{d\left[\frac{p_c(v_2 - v_1)}{MRT}\right]}{dT} + \frac{p_c(v_2 - v_1)}{MRT} \frac{d^2 lp_c}{dT\,dx} = 0.$$

Now $lp_c = lp_x - f\,\frac{T_x - T}{T}$, $\frac{dlp_c}{dx} = \frac{dlp_x}{dx} - \frac{f}{T}\frac{dT_x}{dx}$, $\frac{d^2 lp_c}{dx\,dT} = \frac{f}{T^2}\frac{dT_x}{dx}$.

If we put $\frac{dlp_x}{dx} = \frac{f}{T_1}\frac{dT_x}{dx}$, in which T_1 can have all possible values from $-\infty$ to $+\infty$, then $\frac{dk}{dT} = 0$ may also be written:

$$-\frac{Td\left[\frac{p_c(v_2 - v_1)}{MRT}\right]}{\left[\frac{p_c(v_2 - v_1)}{MRT}\right]dT} = \frac{\frac{1}{T}}{\frac{1}{T_1} - \frac{1}{T}} = \frac{1}{\frac{T}{T_1} - 1}$$

The first member of this equation is always positive, at lower temperatures nearly equal to 0, and at the critical temperature infinitely great. So the second member must also be positive. Or, if this equation is to be satisfied $T_1 < T$, but positive. In all the cases, therefore, in which T_1 is negative, $\frac{dk}{dT}$ cannot become equal to 0, and no reversal of sign takes therefore place in the course of $\frac{\frac{dp}{dx_1}}{\frac{dp_c}{dx}}$ with the temperature.

So the reversal of sign only occurs, when in the equation

$$\frac{dlp_x}{dx} = \frac{f}{T_1}\frac{dT_x}{dx}$$

T_1 lies between 0 and T_x. The two extreme values give for $T_1 = 0$

the value of $\frac{dT_x}{dx} = 0$, and for $T_1 = T_x$ the value of $\frac{dT_x}{T_x dx} = -\frac{1}{6}\frac{db}{bdx}$

so the well-known limits for mixtures for which $\frac{dp_c}{p_c dx}$ can be equal to 0.

For the initial direction of the section normal to the p-axis, the following equation holds:

$$-\frac{1}{T}\left(\frac{dT}{dx_1}\right)_p = \frac{RT}{r}\left\{\frac{x_2-x_1}{x_1}\right\} = \frac{RT}{r}\left\{e^{\frac{u}{RT}\frac{dp_c}{dx}} - 1\right\}$$

and

$$-\frac{1}{T}\left(\frac{dT}{dx_2}\right)_p = \frac{RT}{r}\left\{\frac{x_2-x_1}{x_2}\right\} = \frac{RT}{r}\left\{1-e^{-\frac{u}{RT}\frac{dp_c}{dx}}\right\}$$

Both yield at the critical temperature of the components:

$$-\left(\frac{dT}{Tdx}\right)_p = \frac{RT}{r}\frac{u}{RT}\frac{dp_c}{dx} = \frac{u}{r}\frac{dp_c}{dx} = \frac{\frac{1}{p_c}\frac{dp_c}{dx}}{\frac{T}{p}\frac{dp}{dT}} = \frac{1}{7}\frac{1}{p_c}\left(\frac{dp_c}{dx}\right)_x$$

According to results obtained before, we may also write:

$$-\frac{1}{T}\left(\frac{dT}{dx_x}\right)_p = -\frac{6}{7}\left\{\frac{dT_k}{T_x dx_x} + \frac{1}{6}\frac{1}{b}\frac{db}{dx_x}\right\}.$$

Physics. — "*The exact numerical values for the properties of the plaitpoint line on the side of the components.*" By Prof. VAN DER WAALS.

In my two previous communications, inserted in the proceedings of this meeting, viz. I on the properties of the plaitpoint line on the side of the components and II on the properties of the sections of the surface of saturation on the side of the components, it has again appeared, that the thermodynamic treatment of such problems enables us to find a complete general solution — but also that if we want to compute numerical values in special cases, the knowledge of the equation of state is indispensable. In some cases it will be sufficient, if we make use of an approximate equation of state; but as soon as the density of the substance is comparable to that in the critical state, the numerical values calculated by means of the approximate equation of state can deviate strongly from reality. This is specially the case with quantities which either refer to the volume, or are in close connection with it. Thus it is known, that already the critical volume of a simple substance is not

equal to $3b$, the value furnished by the equation of state, in which b is put constant, but that this equation is found rather nearer to $2b$. This may be accounted for by taking into account that b is variable and decreases with the volume. In a mixture b also depends on the composition. Accordingly the quantity $\frac{db}{dx}$ is an intricate expression for mixtures, and must in general be distinguished from $\left(\frac{db}{dx}\right)_v$. If the way in which b depends on volume and composition, was accurately known, then there would not be left any difficulties but those of toilsome and intricate calculations. But it is sufficiently known, that the way in which b, even for a simple substance, depends on v, has not yet been fixed with perfect certainty, and that in any case the knowledge of the numerical values, which occur in given forms of b, is wanting. These considerations led me to believe that this would be an objection to deriving theoretically the properties of the beginning of the plaitpoint line with perfect certainty — and also to determining the numerical values exactly. It has however, appeared to me that the knowledge of how b depends on x and v is not required for this exact determination; but that for this purpose it suffices to know two quantities which have been experimentally determined for the critical state of a simple substance.

Let us call f the value which $\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_v = \frac{T}{p}\frac{dp}{dT}$ has in the critical conditions of the component, and $\varkappa$, the critical coefficient, so that $MRT_\varkappa = \varkappa (pv)_\varkappa$.

From $p = \frac{MRT}{v-b} - \frac{a}{v^2}$ follows $\frac{MRT}{p(v-b)} = f$ and $\frac{a}{pv^2} = f - 1$.

The equality of $MRT = \varkappa pv = f(v-b)p$, gives the value

$$v = \frac{f}{f-\varkappa} b$$

for the critical volume, in which we have to keep in view, that now that b is put variable with the volume, b represents the value which this quantity has in the critical state. With $f = 7$ and $\varkappa = \frac{15}{4}$ we find $\frac{v}{b} = \frac{28}{13}$, whereas with $f = 4$ and $\varkappa = \frac{8}{3}$ we should find the value $\frac{v}{b} = 3$. For carbonic acid KEESOM has found $f = 6{,}7$ and $\varkappa = 3{,}56$, from which would follow $\frac{v}{b} = \frac{6{,}7}{3{,}14} = 2{,}134$,

If in $MRT = \varkappa pv$ we put the value of v, we find:

$$MRT = pb\frac{\varkappa f}{f-\varkappa}.$$

With $f = 4$ and $\varkappa = \frac{8}{3}$ the factor of $pb = 8$, and with $f = 7$ and $\varkappa = \frac{15}{4}$ this factor is found to be only slightly different viz. $8\frac{1}{13}$. For the calculation of the value of b in the critical condition we get therefore:

$$b = \frac{MRT}{p}\frac{f-\varkappa}{\varkappa f} \quad \text{or} \quad \frac{T_\varkappa}{p_\varkappa}\frac{1}{273}\frac{f-\varkappa}{\varkappa f}.$$

If we put the value of v in the equation $\frac{a}{pv^2} = f-1$, we find:

$$p = \frac{a}{b^2\frac{f^3(f-1)}{(f-\varkappa)^2}}$$

The factor of b^2, which with $f = 4$ and $\varkappa = \frac{8}{3}$ has the well known value of 27, is found slightly above 27,8 with $f = 7$ and $\varkappa = \frac{15}{4}$.

If in $MRT = \varkappa pv$ we substitute the values found for p and v, we find:

$$MRT = \frac{a}{b}\frac{\varkappa(f-\varkappa)}{f(f-1)}.$$

If we again put $f = 4$ and $\varkappa = \frac{8}{3}$, we find $MRT = \frac{8}{27}\frac{a}{b}$; with $f = 7$ and $\varkappa = \frac{15}{4}$ the factor $\frac{a}{b}$ is equal to $\frac{1}{3.446}$; also this value differs but little from $\frac{8}{27} = \frac{1}{3.375}$.

For the calculation of a with the critical values of T and p, the formula:

$$a = \frac{(MRT)^2}{p}\frac{f-1}{\varkappa^2}$$

holds.

The factor $\frac{f-1}{\varkappa^2}$ is equal to $\frac{27}{64} = \frac{1}{2,37}$ with $f = 4$ and $\varkappa = \frac{3}{3}$. With $f = 7$ and $\varkappa = \frac{15}{4}$ it is again only slightly different, viz. $\frac{96}{225} = \frac{1}{2,34}$.

For the critical condition $\left(\frac{\partial p}{\partial v}\right)_T$ must be 0. From this follows:

$$\frac{MRT}{(v-b)^2}\left(1-\frac{\partial b}{\partial v}\right)=2\frac{a}{v^3},$$

and after substitution of the values found for MRT and v

$$1-\frac{\partial b}{\partial v}=2\frac{\varkappa(f-1)}{f^2}.$$

With $f=4$ and $\varkappa=\frac{8}{3}$ it follows naturally that $\frac{\partial b}{\partial v}=0$, whereas with $f=7$ and $\varkappa=\frac{15}{4}$ it follows that:

$$1-\frac{\partial b}{\partial v}=\frac{45}{49}.$$

In the same way $\left(\frac{\partial^2 p}{\partial v^2}\right)_T$ must be 0 in the critical state. From this follows:

$$-b\frac{\partial^2 b}{\partial v^2}=2\frac{\varkappa(f-1)(f-\varkappa)(f-4)}{f^4}.$$

With $f=4$ and $\varkappa=\frac{8}{3}$ this value is of course equal to 0. With $f=7$ and $\varkappa=\frac{15}{4}$ we find:

$$-b\frac{\partial^2 b}{\partial v^2}=0{,}1827\ ^1).$$

Let us now proceed to calculate the value of $\frac{dT}{Tdx}$ at the beginning of the plaitpoint line. We have the formula:

$$\frac{dT}{Tdx_0}=\frac{\left(\frac{\partial^2 p}{\partial v\partial x}\right)_T+\frac{1}{MRT}\left(\frac{\partial p}{\partial x}\right)^2_{vT}}{-\frac{\partial^2\varepsilon}{\partial v^2}},$$

and have therefore to determine $\left(\frac{\partial p}{\partial x}\right)_{vT}$ and $\left(\frac{\partial^2 p}{\partial x\partial v}\right)_T$ for the critical condition, but on the supposition that b varies with the volume and that

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1) This high value of $-b\frac{\partial^2 b}{\partial v^2}$ supports the hypothesis that b, in its dependence on the volume, has a more intricate form than is represented by a series of ascending powers of $\left(\frac{b_\infty}{v}\right)$.

$\frac{db}{dx}$ has different values depending on the variations of the volume.

Now

$$\left(\frac{\partial p}{\partial x}\right)_{vT} = \frac{MRT\left(\frac{\partial b}{\partial x}\right)_v}{(v-b)^2} - \frac{da}{dx}\frac{1}{v^2} = -\frac{a}{v^2}\left\{\frac{1}{a}\frac{da}{dx} - \frac{MRT}{a}\left(\frac{\partial b}{\partial x}\right)_v \frac{v^2}{(v-b)^2}\right\}$$

If we call $\frac{db}{dx}$ the value denoting the change of b with change of x, when we make also the volume vary in such a way that the mixture is again in the state which may be called the critical state of the unsplit mixture, then:

$$\frac{db}{dx} = \left(\frac{\partial b}{\partial x}\right)_v + \frac{\partial b}{\partial v}\left(\frac{dv}{dx}\right)_{kr}$$

and $v_\varkappa$ being $= b\frac{f}{f-\varkappa}$, $\left(\frac{dv}{dx}\right)_\varkappa = \frac{db}{dx}\frac{f}{f-\varkappa}$, when f and $\varkappa$ are constant, which is the case when the law of corresponding states is fulfilled. We find then:

$$\left(\frac{\partial b}{\partial x}\right)_v = \left(1 - \frac{\partial b}{\partial v}\frac{f}{f-\varkappa}\right)\frac{db}{dx}.$$

We have to know:

$$\frac{MRT}{a}\left(\frac{\partial b}{\partial x}\right)_v \frac{v^2}{(v-b)^2} = \frac{MRT}{\frac{a}{b}}\frac{v^2}{(v-b)^2}\left\{\frac{f\left(1-\frac{\partial b}{\partial v}\right)-\varkappa}{f-\varkappa}\right\}\frac{1}{b}\frac{d}{dx}.$$

When we substitute the values found above for MRT, v and $1 - \frac{\partial b}{\partial v}$ we find $\frac{f-2}{f-1}\frac{1}{b}\frac{db}{dx}$ — and so:

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = -\frac{a}{pv^2}\left\{\frac{1}{a}\frac{da}{dx} - \frac{f-2}{f-1}\frac{1}{b}\frac{db}{dx}\right\}$$

or

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = -(f-1)\left\{\frac{dl\frac{a}{b}}{dx} + \frac{1}{f-1}\frac{dlb}{dx}\right\}$$

This value is in a high degree dependent on f.

With $f = 4$ we find $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = -3\left\{\frac{dT_\varkappa}{T_\varkappa dx} + \frac{1}{3}\frac{db}{bdx}\right\}$.

With $f = 7$ on the other hand $\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = -6\left\{\frac{dT_\varkappa}{T_\varkappa dx} + \frac{1}{6}\frac{db}{bdx}\right\}$.

In the preceding communication I have concluded to the same value from the equality of $\left(\frac{\partial p}{\partial x}\right)_v$ and $\frac{dp_c}{dx}$ in the critical circumstances, and by means of the empirical formula $-l\frac{p_c}{p_x} = f\frac{T_x - T}{T}$. For from this formula follows

$$\frac{dp_c}{p_c dx} = \frac{dp_x}{p_x dx} - f\frac{dT_x}{T dx}$$

or

$$\frac{1}{p_x}\frac{dp_c}{dx} = \frac{dT_x}{T_x dx} - \frac{1}{b}\frac{db}{dx} - f\frac{dT_x}{T_x dx}$$

or

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = -(f-1)\left\{\frac{dT_x}{T_x dx} + \frac{1}{f-1}\frac{db}{b dx}\right\}.$$

But we could arrive at this equation in a much simpler way still. From:

$$dp = \left(\frac{\partial p}{\partial x}\right)_{vT} dx + \left(\frac{\partial p}{\partial T}\right)_v dT \qquad \left(\frac{\partial p}{\partial v}_{xT} = 0\right)$$

follows, when dT is put equal to dT_x (taking for dT_x the variation of the critical temperature of the unsplit mixture)

$$\frac{1}{p_x}\frac{dp}{dx} = \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} + \left(\frac{T}{p}\frac{dp}{dT}\right)\frac{dT_x}{T_x dx}$$

and so

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{vT} = \frac{1}{p_x}\frac{dp_x}{dx} - f\frac{dT_x}{T_x dx}.$$

And this equation is not only preferable because it is shorter, but also because it is independent of the circumstance whether the law of corresponding states is applicable or not. The value of f in this derivation is that of the component.

Besides $\left(\frac{\partial p}{\partial x}\right)_{vT}$ we have to determine the value of $\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T$.

For this quantity we find:

$$\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T = -2\frac{MRT}{(v-b)^3}\left(1 - \frac{\partial b}{\partial v}\right)\left(\frac{\partial b}{\partial x}\right)_v + MRT\frac{\frac{\partial^2 b}{\partial x \partial v}}{(v-b)^2} + 2\frac{\frac{da}{dx}}{v^2}$$

or

$$\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T = \frac{2a}{v^3}\left\{\frac{1}{a}\frac{da}{dx} - \frac{MRT}{a}\frac{v^3}{(v-b)^3}\left(1 - \frac{\partial b}{\partial v}\right)\left(\frac{\partial b}{\partial x}\right)_v + \frac{MRT}{2a}\frac{v^3}{(v-b)^2}\frac{\partial^2 b}{\partial x \partial v}\right\}.$$

In this expression only the quantity $\frac{\partial^2 b}{\partial x \partial v}$ is unknown. We determine it from:

$$\left(\frac{\partial b}{\partial x}\right)_v = \left\{1 - \frac{\partial b}{\partial v}\frac{f}{f-\varkappa}\right\}\frac{db}{dx}.$$

From this follows:

$$\frac{\partial^2 b}{\partial x \partial v} = -\,b\,\frac{\partial^2 b}{\partial v^2}\,\frac{f}{f-\varkappa}\left(\frac{1}{b}\,\frac{db}{dx}\right).$$

If we substitute the values given above for MRT, v, $\left(1 - \frac{\partial b}{\partial v}\right)$ and $-\,b\,\frac{\partial^2 b}{\partial v^2}$ in the expression for $\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T$, we find for the value of the second term $-\,\frac{1}{b}\,\frac{db}{dx}\,2\,\frac{(f-2)}{f}$ and for the value of the third term $+\,\frac{1}{b}\,\frac{db}{dx}\,\frac{f-4}{f}$.

The value of $\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T$ is then found equal to:

$$\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T = \frac{2a}{v^3}\left\{\frac{1}{a}\,\frac{da}{dx} - \frac{1}{b}\,\frac{db}{dx}\right\}$$

or

$$\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T = \frac{2a}{v^3}\,\frac{dT_x}{T_x dx},$$

and for $\dfrac{\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T}{-\,\frac{\partial^2 E}{\partial v^2}}$ we find the simple value $\frac{dT_x}{T_x dx}$; so exactly the same value as follows from the equation of state, in which b is put constant. This gives rise to the conjecture that this relation might be found merely from thermo-dynamic relations independent of the knowledge of the equation of state, and this is indeed the case.

Let us consider the quantity $\left(\frac{\partial p}{\partial v}\right)_T$. It is equal to 0 in the critical state of the component. Let us pass from this homogeneous critical phase to another in which the volume has changed with dv, the composition with dx, and the temperature with dT.

Let us put dT again equal to dT_x, so let us assume that the mixture with dx molecules of the second kind is again in an homogeneous critical phase, then $\left(\frac{\partial p}{\partial v}\right)_T$ is again equal to 0.

From:

$$d\left(\frac{\partial p}{\partial v}\right)_T = \left(\frac{\partial^2 p}{\partial v^2}\right)_{Tx} dv + \left(\frac{\partial^2 p}{\partial x \partial v}\right)_T dx + \left(\frac{\partial^2 p}{\partial v \partial T}\right)_x dT_x$$

follows, because $d\left(\frac{\partial p}{\partial v}\right)_T$ and $\left(\frac{\partial^2 p}{\partial v^2}\right)_{Tx}$ are equal to 0:

$$\left(\frac{\partial^2 p}{\partial x \partial v}\right)_T = -\left(\frac{\partial^2 p}{\partial v \partial T}\right)_x \frac{dT_x}{dx},$$

and from the relation:

$$T\left(\frac{\partial p}{\partial T}\right)_v - p = \frac{\partial \varepsilon}{\partial v}$$

follows:

$$T\left(\frac{\partial^2 p}{\partial v \partial T}\right) = \frac{\partial^2 \varepsilon}{\partial v^2}.$$

And from this we find again now only from thermodynamic relations what we have derived already above.

As we found, also by means of mere thermodynamics:

$$\left(\frac{\partial p}{\partial x}\right)_{vT} = -fp\left\{\frac{dT_x}{T_x dx} - \frac{1}{f}\frac{dp_x}{p_x dx}\right\}$$

we may put without making use of the equation of state:

$$\frac{dT}{T dx_0} = \frac{dT_x}{T_x dx} - \frac{f^2 p^2}{MRT\frac{\partial^2 \varepsilon}{\partial v^2}}\left\{\frac{dT_x}{T_x dx} - \frac{1}{f}\frac{dp_x}{p_x dx}\right\}^2.$$

The factor $-\frac{f^2 p^2}{MRT\frac{\partial^2 \varepsilon}{\partial v^2}}$ may be reduced to a simple form, but for the determination of the value of this factor it is required to know the equation of state. If we write $fp = \frac{MRT}{v-b}$, and $\frac{\partial^2 \varepsilon}{\partial v^2} = -2\frac{a}{v^3}$, this factor becomes equal to $\frac{MRT}{(v-b)^2} : \frac{2a}{v^3}$ and as $\frac{MRT}{(v-b)^2}\left(1 - \frac{\partial b}{\partial v}\right) = \frac{2a}{v^2}$ follows from $\left(\frac{\partial p}{\partial v}\right) = 0$, we get:

$$-\frac{f^2 p^2}{MRT\frac{\partial^2 \varepsilon}{\partial v^2}} = \frac{1}{1 - \frac{\partial b}{\partial v}} = \frac{f^2}{2\varkappa(f-1)},$$

so with $f = 7$ and $\varkappa = \frac{15}{4}$ the value of this factor becomes equal to $\frac{49}{45}$. Hence we have:

$$\frac{dT}{T dx_0} = \frac{dT_x}{T_x dx} + \frac{49}{45}\left\{\frac{dT_x}{T_x dx} - \frac{1}{7}\frac{dp_x}{p_x dx}\right\}^2.$$

If we introduce the quantity b instead of $\frac{dp_x}{p_x dx}$, we find:

$$\frac{dT}{Tdx_0} = \frac{dT_x}{T_x dx} + \frac{f-1}{2x}\left\{\frac{dT_x}{T_x dx} + \frac{1}{f-1}\frac{db}{bdx}\right\}^2.$$

With $f = 4$ and $x = \frac{8}{3}$ we find again $\frac{f-1}{2x} = \frac{9}{16}$, but with $f = 7$ and $x = \frac{15}{4}$, $\frac{f-1}{2x}$ rises to 0,8. With $f = 6{,}7$ and $x = 3{,}56$ (Keesom's values for carbonic acid) the value is not appreciably different from 0,8. If we calculate with $\frac{dT_x}{T_x dx} = -0{,}493$ and $\frac{1}{b}\frac{db}{dx} = -0{,}271$, $f = 6{,}7$ and $x = 3{,}56$ the value of $\frac{dT}{Tdx_0}$, we find for this value $-0{,}259$. Though 0,259 is smaller than the values calculated from Keesom's observations, 0,284 for $x = 0{,}1047$ and 0,304 for $x = 0{,}1994$, we must not forget that the calculated value would hold for the limiting case, viz $x = 0$; and the fact that for $\Delta x = 0$ a smaller value than 0,284 would have to be expected is at least in harmony with the circumstance that the amount is found higher for a higher value of x.

It is evident from all this that though we cannot do quite without the equation of state for the calculation of $\frac{dT}{Tdx}$ for the plaitpoint line, yet it is not necessary to know the form of the quantity b.

For the calculation of the quantity $\frac{T}{p}\frac{dp}{dT}$ for the beginning of the plaitpoint line we have from the formula:

$$dp = \left(\frac{\partial p}{\partial x}\right)_{vT} dx + \left(\frac{\partial p}{\partial T}\right)_{vx} dT$$

the relation:

$$\left(\frac{T}{p}\frac{dp}{dT}\right)_{pl} = \frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_v + \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_{Tv}\frac{Tdx_0}{dT}$$

or

$$\left(\frac{T}{p}\frac{dp}{dT}\right)_{pl} = f + \frac{-(f-1)\left\{\frac{dT_x}{T_x dx} + \frac{1}{(f-1)}\frac{1}{b}\frac{db}{dx}\right\}}{\frac{dT_x}{T_x dx} + \frac{f-1}{2x}\left\{\frac{dT_x}{T_x dx} + \frac{1}{(f-1)}\frac{1}{b}\frac{db}{dx}\right\}^2}$$

or in numerical value for the mixture of oxygen and carbonic acid:

$$\left(\frac{T}{p}\frac{dp}{dT}\right)_{pl} = 6{,}7 + \frac{-5{,}7\{-0{,}493 - 0{,}047\}}{-0{,}259} = 6{,}7 - 11{,}28 = -4{,}58$$

With the mixture $x = 0{,}1047$ KEESOM has found $-6{,}3$ and with $x = 0{,}1995$ the amount found was $-6{,}08$.

If we take the product of $\frac{T}{p}\frac{dp}{dT}$ and $\frac{dT}{Tdx}$ we find the value of $\frac{1}{p}\frac{dp}{dx}$ for the beginning of the plaitpoint line. As both $\frac{T}{p}\frac{dp}{dT}$ and $\frac{dT}{Tdx}$ are negative for the mixture of carbonic acid and oxygen, the value of $\left(\frac{1}{p}\frac{dp}{dx}\right)_{pl}$ is positive.

Anatomy. — "BOLK's *centra in the cerebellum of the mammalia*". By D. J. HULSHOFF POL (from the laboratory for Psychiatry and Neurology at Amsterdam). (Communicated by Prof. C. WINKLER).

In his well-known researches about the cerebella of mammalia [1]), BOLK concludes: that "the Lobus anterior cerebelli does contain the centre of coordination for the muscle-groups of the head, the Lobus simplex the centrum of coordination for those of the neck; the non-symmetrical centre of coordination for both left and right extremity is situated in the Lobus medianus posterior, whilst each of the Lobuli ansiformes is the seat of one of the symmetrical centra, respectively for both right, and for both left extremities." [2])

Within the same line of research, VAN RIJNBERK [3]) at LUCIANI's laboratory in Roma, experimenting on two dogs, extirpated a portion of the cerebellum, with the aim of taking away the right part of the Lobus simplex.

The secondary symptoms, which were observed during the first days after the operation, having passed away, the animal experimented upon continued shaking its head, as if it meant to say "no".

This symptom resembled very much a trouble in the coordination, and such being indeed the case, it would have confirmed the hypothesis of BOLK. Therefore it was important to determine with as much exactness as possible, which portion of the cerebellum had been removed.

To this purpose the preparation, fixed in formol, was offered

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1) Prof. Dr. L. BOLK. Das Cerebellum der Säugetiere. PETRUS CAMPER, Vol III, part. I. Amsterdam.

2) Prof. Dr. L. BOLK. Over de physiologische beteekenis van het cerebellum. De Erven BOHN, Haarlem. 1903.

3) G. A. VAN RIJNBERK. Tentative di localisazione funzionali nel cerveletto. Archivio di fisiologia. Vol. I. Fasc. V.

for examination to Professor WINKLER, who had the kindness to leave its further elaboration to me.

Since the sections, that should be made subsequently, were to be stained after the WEIGERT—PAL method, the cerebellum was immediately after its arrival in Amsterdam refixed in MULLER's liquid. It was only when this had been performed that the photographs were taken (fig. I and II).

The white spots that are seen on the figures, were caused by celloidine, by means of which the pieces were pasted together. It was necessary to do this, because the cerebellum, was received here being cut into three pieces.

In the middle of the surface of the cerebellum we observe a cavity. If this cavity is divided into four parts along the longitudinal axis of the cerebellum, nearly one quarter is lying to the left of the median line, two other quarters are lying in the right median part, and another quarter (probably the smallest one) is lying in the right lateral part.

The form of this cavity on the surface of the cerebellum, as far as it is lying in the median portion, is nearly that of a truncated isosceles triangle having for its basis the paramedian line.

The greater part of this triangle (nearly three quarters) is lying in the right half, and only one fourth in the left half of the cerebellum.

What imports most now is to find out to which subdivision of the cerebellum belong the convolutions from which van RIJNBERK has extirpated this small piece.

In fig. I and II, next to the defect (fig. II sub 1), our attention is drawn immediately by a deep furrow (fig. II sub 2*a*), that has become to all probability more clearly visible by the process of fixation, than may have been the case during life.

The sulcus primarius is the furrow penetrating deepest to the medullary nucleus and continuing forward till near the sinus Rhomboidalis, causing in this way the lobus anterior and the lobus posterior to be connected, for by far the greater part, only by a ridge of medullated nerve-substance. We may therefore safely assume that, the cerebellum having been eventually shrivelled, this sulcus, lying between two portions so deeply divided, will become more distinctly visible.

At first view therefore we might hold the furrow indicated sub 2*a* fig. II, to be probably the sulcus primarius. Such being the case, all that is lying before this furrow would be lobus anterior, all that is lying behind it lobus posterior.

On examining the anterior portion, this is found to consist of two

parts, that may be discriminated with sufficient distinctness (fig. II, 3 and 4). What we find indicated sub 3 is a coniform swelling, consisting of a succession of folia, separated by sulci running in the direction of the margo mesencephalicus.

Accordingly it does not offer any difficulty to recognise in this portion the lobus anterior.

The case is different however for the folium behind this part (fig. II sub 4) lying before the sulcus mentioned sub 2*a*.

It would belong to the anterior lobe if this furrow were indeed the sulcus primarius; but as it is lying behind the sulcus sub 2*b*, the direction of which is totally different from that of the other sulci of the lobus anterior, the question arises whether this convolution sub 4 does indeed belong to the anterior lobe.

The direction of this gyrus is totally different from that of all the other convolutions lying before it, because it does encompass the basis of the coniform swelling. Whilst the convolutions in the anterior part are ranged regularly behind one another, the convolution sub 4 does diverge from that arrangement, because the former convolutions are implanted in this latter. Relying only on the difference in direction between these convolutions, one would be inclined to consider as the sulcus primarius rather the sulcus sub 2*b* than that sub 2*a*.

We see however, that the convolution sub 4, like that of the coniform swelling, is running from the right to the left. It is uninterrupted and the initial direction of this curved convolution is likewise towards the margo mesencephalicus. This — in addition to the fact, that Bolk in his description of the cerebella of different mammalia, likewise reckons the lower and more deviating convolutions to the lobus anterior — supports the opinion that the sulcus sub 2*b* is not the sulcus primarius, as we might suppose, if relying only on the difference in direction between the convolutions sub 3 and 4.

The macroscopical description will therefore have to leave undecided the question, whether the convolution sub 4 must be reckoned to the lobus anterior or to the lobus posterior.

Nevertheless it is of the greatest importance to delimitate exactly to which portion of the cerebellum this convolution belongs, because it has become evident from the figures I and II, that on the surface it is precisely in this convolution that the greater part of the defect is situated. The examination of sagittal sections of the cerebellum will have to decide this question.

All that is lying behind the anterior lobe belongs to the posterior

lobe. This posterior part, with the exception of its first convolutions, is divided into one median and two lateral portions by the sulci paramediani, which run parallel to the median line.

Consequently all that is lying between the two paramedian sulci forms the median part of the lobus posterior, all that is lying to the right and to the left of them forms the lateral part of this lobe.

In figure II sub 11 we find the sulcus intercruralis. This furrow is lying in the middle of the lobus ansiformis, (fig. I). The convolutions that start originally from the median line as crus primum (fig. II sub 9) and bend gradually, when arrived in the ultimate lateral part, to return thence as crus secundum (fig. II sub 10) to the median part, take before reaching it another bent, this time straight backward; to continue further as lobus paramedianus (fig. II sub 12) parallel to the sulcus paramedianus.

In describing further the lobus posterior we will confine ourselves, for the sake of convenience, to the left half.

Of course in fig. I and II all that is lying to the left of the sulcus paramedianus belongs to the lateral part, and consequently may not be reckoned to the lobus simplex, the convolutions of this latter, according to Bolk, continuing without any interruption from the right to the left.

Applying this test to fig. I, we find that the extreme end of the left sulcus paramedianus is stopped by a convolution indicated sub 5 fig. II.

Thence it might be concluded, that the convolutions indicated sub 5 and 6 in fig. II, accordingly lying above the sulci paramediani and below the lobus anterior, form the lobus simplex.

On a closer examination of the lowest of these two gyri, i. e. the gyrus sub 5, we find however, that to the left of the white spot (the end of the dotted line), in the lateral part of the gyrus therefore, a narrow furrow may still be observed, that does not continue to the median line. According to Bolk therefore, this convolution does not belong to the lobus simplex, as the incomplete furrows in this lobe, like those in the lobus anterior, ought to start from the median line.

The cause, why the interruption of this gyrus sub 5 is not, as usually, visible on the surface, must be sought in the fact that the sulcus paramedianus disappears in the depth, and does not therefore penetrate into this convolution on the surface.

The last convolution, sub 6, fulfills in every respect the conditions claimed by Bolk for the convolutions of the lobus simplex; as it

lies directly behind the lobus anterior, and continues without interruption from the right to the left, whilst incomplete furrows, not starting from the median line, do not occur in it. Moreover the convolutions forming the crus primum, lie adjacent to it and originate in it.

This convolution sub 6 thus forming the lobus simplex on the surface, does not continue very far on the more lateral part, as may be seen in the figure. It is therefore little developed.

Now if we follow to the right the course of the convolution sub 4 fig. II, about which it is not yet decided whether it belongs to the anterior or to the posterior lobe, we find that this convolution loses itself in the cavity sub 1. It may be said therefore that the operation has extirpated at least on the surface, a part of the left median portion and the whole of the right median portion of this gyrus.

I have stated already that the cavity broadened to the right, and attained its largest breadth in the prolongation of the sulcus paramedianus. By the photograph sub I it becomes evident, that in this place the lesion extends over the convolutions lying before and behind.

A convolution of the lobus anterior lying before the convolution sub 4, and a convolution of the lobus simplex behind it, we may therefore assume, in as much as it is allowed to draw conclusions from the macroscopical aspect, that in the right median part at least one convolution of the lobus anterior and one of the lobus simplex have been injured. It must remain undecided whether the principal defect, situated in the convolution sub 4 fig. II, ought to be reckoned to the lobus anterior or to the lobus simplex.

It was on purpose I did not hitherto say anything about the macroscopical deviations in the right lateral portion, because, as was stated before, the whole of it was divided from the left portion of the cerebellum and having been thrivelled in the course of the elaboration, it no longer fitted exactly unto the median part, as may distinctly be seen in fig. I and II.

In order therefore to avoid eventual errors I neglect the macroscopical description of the lateral portion of the posterior lobe. This omission does not involve unsurmountable difficulties, because the description of the sagittal sections remains still to be given, and by means of these latter we shall have to find out which portions have been destroyed and which have been left intact.

The cerebellum having been fixed for some time at the laboratory in MULLER's liquid it was inclosed in celloidine and cut in serial sections.

Photograph III has been taken of a section on the left border of the defect, i.e. on the spot where the lesion begins on the left.

Photograph IV has been taken of a section from the right median part, directly adjacent to the median line.

Photograph V represents a section from the middle of the median portion.

Photograph VI represents a section very close to the prolongation of the sulcus paramedianus dexter, but still within the median portion.

Photograph VIII represents a section from the left lateral, being the un-injured portion. It corresponds with the place in the right lateral part, represented by photograph VII.

If, by the aid of photograph III, we try to delimitate the exact situation, especially of the different convolutions around the sulcus primarius, it does not present any great difficulty to know which is the lobus anterior and which the lobus posterior.

The furrow, lying opposite the sinus Rhomboidales (R.), is the sulcus primarius (s. p.). All that is lying before this sulcus, to the left of it in fig. III, belongs to the lobus anterior, all that is lying behind it, to the right in the figure, belongs to the lobus posterior.

The strongly developed anterior lobe is divided into four lower lobules, which I have indicated sub 1, 2, 3 and 4, conform to BOLK's description.

Accordingly these numbers correspond with the lobes, designated in the human cerebellum as Lingula, Lobus centralis and Culmen.

For the posterior lobe I likewise followed BOLK's division, and accordingly designated the folia by *a*, *b*, *c* and *d*; *a* corresponding with Nodulus, *b* with Uvula, *c* with Pyramis and *d* with Tuber vermis, Folium cacuminis and Declive. This latter would be the Lobus simplex.

The rationality of BOLK's division is demonstrated clearly by this preparation, as the medullary rays of the folia are all of them separately implanted in the medullary nucleus.

The sinus Rhomboidales, the roof of the fourth ventricle, is designated sub R. Opposite to it, accordingly in the figure straight above it, and separated from it only by the medullary nucleus, we find the sulcus primarius (s. p.).

As we stated before, it could not be decided with any certainty from the macroscopial description whether the sulcus primarius was to be sought for sub 2*a* or sub 2*b* (fig. II), and consequently the situation of the defect could not be precisely defined; it is therefore necessary to determine with the utmost exactness in their mutual

relation the respective situations of the sulcus primarius, the adjacent gyri and the lesion.

To this purpose I have designated in fig. III sub α and sub β the two convolutions lying next to the sulcus primarius on the surface.

Looking ad α, which represents the first convolution of the lobus posterior on the surface, we find that it consists of a secondary radius medullaris ending in a bifurcation on the surface. Such not being the case with the adjacent secondary medullary rays of c_3, this convolution may be likewise easily recognized in the next figures.

The same thing may be said for β, which represents the most posterior convolution of the lobus anterior on the surface. For we observe that the medullary ray of the lobule N°. 4 divides itself into two portions: the posterior one β being the prolongation of the thick primary radius medullaris and therefore easily recognised.

In photograph III, representing, as may be remembered, a section taken from the place where to the left the lesion begins, we see clearly that not the entire convolution β has been destroyed, but mainly that portion of it that is lying next to the sulcus primarius. The convolution of the lobus posterior, lying behind it, has not been injured at all, its surface, on the spot where this convolution bends inward towards the sulcus primarius being distinctly visible.

This spot (α) being of importance in order to determine whether the lobus posterior, in case the lobus simplex, has been injured, I have designated it likewise on the other photographs (IV, V and VI). Anticipating for a moment on the subsequent description of these photographs, I may state that they show clearly that this spot on the surface, where the convolution bends inward, presents nowhere any trace of lesion.

The direct conclusion to be derived from this fact is that the lobus simplex has not been injured AT ITS SURFACE.

As to the convolution β however matters stand differently.

In fig. III already we may see that from 4, representing the posterior folium of the lobus anterior, the posterior secondary convolution β in the upper part has been almost entirely destroyed. Only a small piece of its most anterior portion remains.

In the direction of the medullary nucleus the lesion extends only over the upper third part of the sulcus primarius.

The secondary radius medullaris is still distinctly visible at the spot where it is united to the anterior convolution.

Surveying the successive aspects of the lesion in the figures IV, V and VI, we find that in IV a very small remnant of the convolution sub β still subsists, whilst the secondary radius medul-

laris has been cut off almost up to the place of bifurcation of the primary radius.

The lesion itself penetrates inward, a little into the medullary nucleus, moreover the secondary and tertiary lobules, lying adjacent to the sulcus primarius, are for the greater part, if not wholly, destroyed.

This becomes still more evident from the fig. V and VI, where all that belongs to the convolution β, has been destroyed. The lesion itself penetrates still deeper, in fig. V it has nearly cleft the radius medullaris, in fig. VI it has done so entirely.

We may therefore conclude: *that in the median portion, more especially in its right half, the posterior lobule of the lobus anterior has been seriously injured, that even nearly the whole of it has been removed.*

I stated already, that from the anterior portion of the lobus posterior, i. e. the lobus simplex, nothing has been destroyed on the surface, as the place where it bends inward sub α, remains visible on all sections in fig. III, IV, V and VI. Deeper however, the case becomes different.

In fig. IV we observe that all secondary lobules, lying adjacent to the sulcus primarius in the depth, have been completely destroyed. In fig. V there has been removed still more, nearly the whole of the secondary radius medullaris having been extirpated. In fig. VI it is entirely destroyed, whilst moreover the primary radius medullaris of the small lobe c_2 has been completely divided from the medullary nucleus.

We may thence conclude that, *though the lobus simplex in its median portion is not injured at its surface, on the contrary in the depth, in the portion adjacent to the sulcus primarius, it has been entirely destroyed, even those convolutions that remained intact on the surface in the paramedian area (figure III sub α) having been divided from the primary radius medullaris.*

Considering next the lateral portion, fig. VIII enables us to survey the situation and the division of the folia under ordinary circumstances. The sulcus primarius (s. p.) still subsists, as the convolution sub 4 fig. II, the last convolution of the lobus anterior, is removed considerably sideways. All that lies before this sulcus, accordingly to the left in fig VIII, belongs to the lobus anterior. Consequently the small lobe sub 1 is the last folium of this lobe. All that lies behind the sulcus primarius, thus belongs again to the lobus simplex sub 2.

Considering next fig. VII, it is shown thereby that on both sides

of the sulcus primarius the secondary lobules have been all destroyed, and moreover from the lobus anterior even nearly the whole of the radius medullaris.

Accordingly we find for the *lateral portion* the same result as for the median right half, i. e. *that besides the greater part of the lobus simplex, also a part of the lobus anterior has been destroyed.*

Originally we intended not only to find out by means of the sections, which portion of the cerebellum had been taken away by VAN RIJNBERK, but likewise to demonstrate the microscopical changes subsequent to the lesion. To this purpose we used the WEIGERT-PAL method of staining.

Unfortunately however, on microscopical examination, it was shown that nearly the whole mass of medulla had taken a granular aspect. It was therefore impossible to study the nerve-fibres, and any secondary degeneration they might have suffered with the method of WEIGERT-PAL, and for MARCHI-preparation the cerebellum proved unfit.

Nevertheless one fact remains worthy of attention: in the part of fig. VI, designated sub *a*, accordingly in that portion, separated by the operation from the central medullaris originating in it, the radius medullaris not only is stained black as distinctly as the other secondary medullary rays, but moreover the PURKINJE corpuscles and their ramifications in this portion (stained by means of osmium acid), do not present any changes worth mentioning, if compared to those of the other, un-injured lobules.

SUMMARY.

A. According to BOLK's theory, the Lobus simplex is the seat of an unsymmetrical centrum of coördination for the muscle-groups of the neck.

B. Operating on a dog, VAN RIJNBEEK extirpated a part of the cerebellum, about the Lobus simplex. In consequence of this operation, its secondary symptoms having passed away, the animal retained a continual movement of the head as if it meant to say "no".

C. Investigations at the laboratory in Amsterdam taught us that the operation had destroyed:

a. in the left median part, next to the median line, a small superficial portion of the last gyrus of the Lobus anterior;

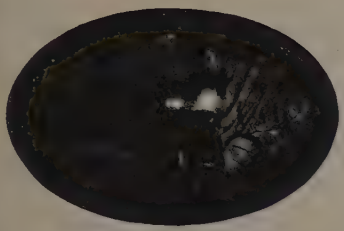
b. between the median line and the paramedian line to the right;

1st. nearly the whole of the last gyrus of the Lobus anterior;

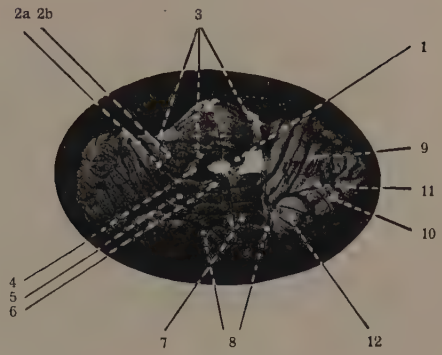
2nd. nothing from the Lobus simplex at its surface;

J. HULSHOFF POL. BOLK's centra in the cerebellum of the mammalia."

I



II



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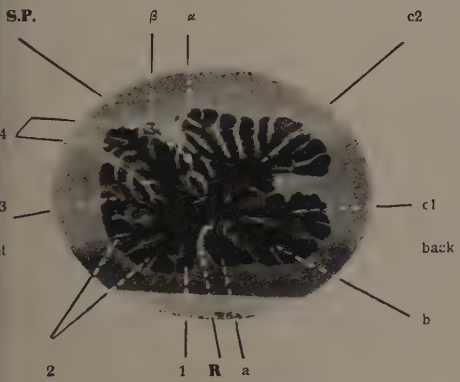
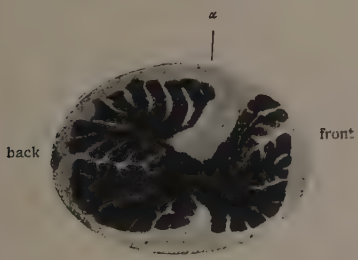


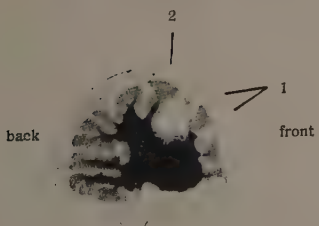
IV

V

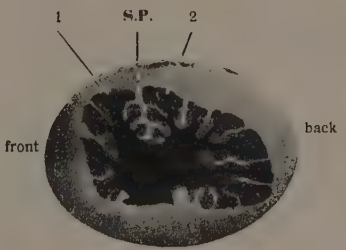


VI

VII



VIII



3rd, nearly the whole of the Lobus simplex in the depth, whilst, towards the paramedian line, likewise those portions of gyri, which remained intact on the surface, have been divided from the primary radius medullaris.

c. In the part, situated to the right of the paramedian line, as far as the lesion extends;

1st. nearly the whole of the posterior folium of the Lobus anterior;

2nd. the greater part of the Lobus simplex.

Physiology. — "*The designs on the skin of the vertebrates, considered in their connection with the theory of segmentation.*" By Dr. G. VAN RIJNBERK. (Communicated by Prof. C. WINKLER).

That there exists some connection between the distribution of pigments in the skin and its segmental innervation will be evident to any one who has made some investigations into the questions concerning the theory of segmentation. Different authors have made numerous unconnected researches about this subject. SHERRINGTON [1]) has pointed out that the stripes of the zebra are ranged in segments on neck and trunk; whilst he identifies the cross-stripe over the shoulders of the ass with its dorsal axis-line for the anterior extremity. WINKLER [2]) has drawn attention to the fact that deep-coloured rabbits often show white spots, presenting a marked conformity in distribution and extension with the analgetic areas that are produced when one or two of the posterior roots of the spinal nerves have been cut through. It may therefore readily be assumed that these white spots find their origin in the fact that either one or several segments lack the faculty of producing pigment. ALLEN [3]) has demonstrated that certain series of spots on the skin of the squirrel correspond with the points of entrance into the hypodermis of series of skinbranches of the intercostal and other homologous nerves. Two

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[1]) C. S. SHERRINGTON, Experiments in examination of the peripheral distribution of the fibres of the posterior roots of some spinal nerves. Philosoph. Transactions of the Royal Society. London, vol. 184 B. p. 757.

[2]) C. WINKLER, Ueber die Rumpfdermatome. Monatschrift für Psychiatrie und Neurologie. Bd. XIII, 1903, h. 3, S. 173.

[3]) H. ALLEN, The distribution of the colour-marks of the mammalia. Proceedings of the Academy of Nat. Sciences of Philadelphia, 1888, p. 84 et seq. — See also, Science. 1887.

years ago I myself [1]) collected some data, by means of which I endeavoured to justify the hypothesis, that in two species of sharks (*Scyllium Catulus* and *Sc. Canicula*) the deep-coloured transversal stripes correspond alternately with groups of five and of three segments, producing more pigment than the other segments.

My present endeavour will be to demonstrate some systematical views concerning the colourmarks on the skin of the vertebrates (birds excepted), on the basis of the rather extensive and detailed knowledge we possess about the segmental innervation of the skin.

To that purpose the first thing to be done is to define as clearly as possible the object of my investigations. In every animal the so-called "design" in its widest acceptation, originates in the contrasting effect of at least two colours or tints. Generally the next proceeding is to select one of these colours, that is then regarded as the "design" in a narrower sense, whilst the other colour is called the prime-colour. This choice is principally determined by esthetic motives: its criterion being either the difference in extension, — the less extensive colour being then taken for the design —, or else the difference in tint, the lightest tint being then regarded as the prime-colour. The irrationality of this method is evident, as has already been pointed out partially by J. Zenneck [2]) a disciple of Eimer. For when comparing together a few cartoons of equal size, on which are designed respectively: a small black figure with a large white margin, a small white figure with a large black margin, a large black figure with a narrow white margin and a large white figure with a narrow black margin, — nobody will think in either of these cases of taking the margin for the design: the figure remains the figure, whether it be large or small, black or white. Consequently it is neither by its tint, nor by its extension that the "design" ought to be distinguished, but only and exclusively by its significance. Applying this test to the distribution of colourmarks on the skin of animals (the design in its widest acceptation) we will accordingly have to determine at the outset in each case, which biological, morphological and physiological significance must be ascribed to the different portions of the design.

Their biological significance may be neglected here; necessarily the

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[1]) G. van Rijnberk, Beobachtungen über die Pigmentation der Haut von Scyllium Catulus und Canicula und ihre Zuordnung zu der segmentalen Hautinnervation dieser Thiere. — Petrus Camper, Vol. III, 1904, part. 1, p. 137—173.

[2]) J. Zenneck, Die Zeichnung der Boiden. Zeitschrift f. Wissenschaftliche Zoologie, Bd. 64, 1896, h. 1, u. 2, S. 234.

foremost condition to obtain correct notions concerning the pigmentation of the skin is to understand properly the morphological framework whereon the design is founded, and the manner in which it is physiologically determined. In endeavouring to elucidate these questions, it becomes evident that the simple distinction between "design" in a narrower sense and "prime-colour", is not sufficient for a rational description of the manifold pigmentations of the skin.

According to my opinion, we ought at least to distinguish three elements, constituting in their complete or partial combination the "design" in its widest acceptation. In order to obtain a proper distinction between these three elements, it is necessary to introduce a quantitative criterion into the problem. Besides the respective plus and minus in the pigmentation, we shall therefore have to make still another distinction, by opposing to the prime-colour respectively an excedent and a defect contrast in the production of pigment.

A few instances may suffice to elucidate this. In a white dog with black ears black represents the contrasting colour just as white does in a black horse with a white mark on its forehead. But in the first case it may be called an excedent contrast, in the second case a defect contrast. In an animal where the prevailing colour is brown showing both black and white marks, we find combined the three elements that ought to be distinguished: the brown prime-colour, the excedent and the defect-contrasts. Starting from these simple instances, we shall be able to compose a complete terminology, by means of which the most important elements of the pigmentation of the skin may be defined with tolerable exactness as to their shape, extension and distribution. For instance the white mark on the forehead of the black horse we will call an isolated defect contrast. Thus the dark stripes on neck and trunk of the *zebra* may be called regular serial diagonal excedent contrasts, whilst the stripes on *Galidictis* are regular serial longitudinal excedent contrasts. The morphological and physiological basis for distinguishing between excedent and defect-contrast, consists in the following points:

1. In a large series of cases excedent-contrasts are found in such places where the innervation of the skin is likewise strongest, whilst on the contrary defect-contrasts are found in such places where this innervation is feeblest. 2. We may observe that the excedent-contrasts often correspond as to their shape and distribution with the caricatures of the dermatoma [1]), whilst the defect contrasts often correspond

[1]) C. Winkler and G. van Rijnberk, Structure and function of the trunk-dermatoma III. These Proc. IV, p. 509. Verslagen der Kon. Akademie v. Wetenschappen, 22 Febr. 1902.

with the analgetic areas called into existence by the destruction of sensibility in one or more segments.

A few instances may serve to illustrate this. The sensibility of the skin under normal circumstances is strongest within a system of lines and zones, corresponding with the average limits of the dermatoma (of more precise *limits* we cannot speak because of the overlapses). This has been proved clinically for man by LANGELAAN [1]), experimentally for the dog by WINKLER [2]) and myself. Now if we observe the dark stripes of the zebra, we find that beyond any doubt these stripes, at least on neck and trunk, show a marked accordance in their distribution and direction with the average limits of the dermatoma, as these latter may be imagined to be, relying on the data procured by PEYER [3]), SHERRINGTON [4]), TÜRCK [5]), WINKLER [6]) and myself respectively for the rabbit, the monkey and the dog.

Their number corresponds nearly with that of the segments of neck and trunk; on the neck and on the trunk they are somewhat wider apart than at the points of insertion of the extremities, this being in perfect accordance with the fact demonstrated by WINKLER and myself that at those points the ranks of the dermatoma are more thickly set. The distribution of the stripes on the extremities is not so easily explained. On superficial view they resemble rings, running around the extremity. In reality however each of these rings consists of two symmetrical semicircles, passing by pairs into one another by a definite angle on the outside and on the inside of the extremity. Connecting together the different points at which the semicircles meet, we obtain two lines corresponding with the dorsal and the ventral axis-lines of the extremity. The direction in which the stripes run (in a caudal direction from the axis-lines), corresponds with

[1]) J. W. LANGELAAN, On the determination of sensory spinal skinfields in healthy individuals. These Proc. III, p. 251.

[2]) See note 1 on the preceding page.

[3]) J. PEYER, Ueber die peripheren Endigungen der motorischen und sensibelen Fasern des Plexus bracchialis. Zeitschrift f. rat. Medizin. N. 7, Bd. IV, 1854, S. 52.

[4]) C. S. SHERRINGTON, loco citato, and: Idem II Ibidem vol. 190, B. 1898, p. 45—186.

[5]) L. TÜRCK, Vorläufige Ergebnisse von Experimental Untersuchungen zur Ermittelung der Hautsensibilitätsbezirke der einzelnen Rückenmarksnervenpaare. Sitzungsber. der K. K. Akad. der Wissensch. zu Wien 1856, and: Die Hautsensibilitätsbezirke der einzelne Rückenmarksnervenpaare. Aus dem literarischen Nachlasse von weil. Prof. Dr. L. TÜRCK zusammengestellt von Prof. Dr. C. WEDL. Denkschriften der Math. Naturw. Classe der K. Akad. der Wissensch. zu Wien. Bd XXIX, 1869.

[6]) C. WINKLER and G. VAN RIJNBERK, On function and structure of the trunk-dermatoma I. These Proc. Vol. IV, p. 266; II. l. c. p. 308; III. l. c. p. 509; IV. l. c. Vol. VI, p. 347.

that followed by the limitlines of the segments of the skin. The number of the stripes, however, is greater than that of the segments can possibly be. But for this difficulty too a solution can be found. On considering the curve of sensibility of a normal trunk-skin, as it has been constructed by WINKLER and myself on the basis of our experiments, we find that at the dorsal median line, where the central areas overlap one another on an average for one third and the dermatomata for one half, a top of the curve i. e. a zone of summation corresponds with every average limit-line between two dermatomata. If now the overlapses amount to more than one half, as they do on the extremities, the curve of sensibility will be much more complicated and the zones of summation therefore more numerous. Accordingly the dark stripes on the extremities correspond likewise with the average lines of demarcation of the dermatomata. In the zebra the excedent of pigment apparently is distributed in accordance with the scheme of the intersegmental zones of summation, and the design resulting from this distribution may therefore be defined as consisting of intersegmental excedent-contrasts. Although this instance may not be entirely isolated, still it is a rather rare one. In many other cases we find that the excedent of pigment is not distributed in accordance with the uniform scheme of intersegmental demarcation, but arbitrarily accumulated in certain points or portions of the segments themselves. A large number of white domestic animals for instance present black spots, showing a marked similitude in their shape, distribution and extension with the figures, denominated by WINKLER caricatures of the dermatomata. The way in which the pigment is distributed, offers even an indication of that peculiar significance which the point of entrance of the skin-nerve apparently possesses for the innervation (maximum and ultimum moriens of the central areas, of the dermatomata and of the sensible skin-areas in general [1]). Thus the series of black dots in many species of sharks, amphibians, serpents and saurias, apparently correspond nearly with the serial ranging of the points of entrance of the dorsal and lateral nerve branches.

We will now turn to the defect contrasts. In deepcoloured specimina of our domestic animals white-tipped ears or tail, a white belly, or a white mark in the frontal median line of the head, or else white toes, are frequently to be met with. It needs not being demon-

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[1]) On function and etc. These Proc. Vol. VI, p. 347. G. VAN RIJNBERK: On the fact of sensible skin areas dying away in a centripetal direction. These Proc. Vol. VI, p. 346.

strated that such marks represent either absolute or relative eccentrical areas. Consequently I consider these marks to be eccentrical defect-contrasts. The white marks in rabbits, to which attention has been drawn by WINKLER, are of a very different nature, being expressions of segmental variability; in the series of equivalent segments, producing pigment, one or two have lost this faculty; thence results the defect, corresponding in shape, distribution and extension with the segmental analgetic areas. A further instance may be forwarded by the so-called Lakenveld cows, whose white "cloth-covering" around the trunk corresponds evidently with a series of pigment-less segments, which have become hereditary by artificial selection in breeding. The above-mentioned white feet may be reckoned likewise to these instances. In black dogs, horses or rabbits, white forefeet and a white mark on the breast are frequently to be met with. Evidently these mean something more than a simple eccentrical defect. It cannot be doubted that such cases represent phenomena of segmental omission.

It is known by the experiments of WINKLER [1]) and myself that the most eccentric skin-segments of the fore-feet (7th and 8th cervical roots), consist only of the lateral portions of the dermatoma, the dorsal parts having entirely vanished whilst the ventral parts are lying exceedingly reduced at the ventral median line near the *manubrium sterni*. Accordingly this relation corresponds perfectly with that of the above described defect areas. For this reason I consider these latter ones to be segmental defect contrasts; they are the expression of a segmental defect-varibility in the 7th and 8th cervical segment. Analogous cases are not rarely found. Frequently the white areas are so extensive that eventually a defect of the 5th and 6th cervical segment may be assumed besides that of the 7th and 8th. Analogous relations exist in the posterior extremity, though we know less about its segmental innervation.

I cannot possibly in these pages enter into more minute details concerning the question of the segmental distribution of the colour marks in the skin. An extensive essay on this subject is shortly to be published. The preceding explanations will however be of sufficient aid to form a judgment concerning my fundamental views and to understand the conclusions, stated in the following summary. Doubtless these conclusions may have some importance for clinical work, because they prove beyond any doubt the great significance of the segmental innervation for the trophic condition of the skin, and add

[1]) C. WINKLER and G. VAN RIJNBERK, Something concerning the growth of the areas of the trunk-dermatoma on the caudal portion of the upper extremity. These Proc. VI, p. 392.

a new support to the probability of the hypothesis that a segmental basis lies at the root of many pathological states, as *naevus pigmentosus* etc.

CONCLUSIONS.

1. The distribution of the pigmentation on the skin of vertebrated animals is in a large series of cases the expression of peculiar relations in the segmental innervation of the skin.

2. In the "skin-design" taken in its widest acceptation, three elements ought to be distinguished: the prime-colour, the excedent-contrast and the defect-contrast.

3. In animals, whose skin is nearly wholly of one colour, the excedent-contrast may be *zonal* (dorsal) or isolated.

An isolated contrast frequently corresponds:

for the head

a. with a definite central nerve-area: (the excedent-contrast in the *N. trigeminus*), or else with definite portions of these areas (Point of entrance of the nerve in the hypodermis; the excedent-contrast *ex introitu;* the supraorbital mark).

for the remainder of the body:

b. with definite isolated skin-segments, more pigmented than the other segments, or with definite sub-divisions of these segments (caricatures of the dermatoma; segmental excedent variability; segmental excedent contrast).

c. with zones of intersegmental summation (intersegmental excedent contrast; the cross on the back of the ass).

4. The defect contrast in animals that are nearly wholly of one colour may appear as a lack of this colour either *zonal* (ventral) or *isolated.* The isolated defect contrast frequently corresponds with:

a. definite nerve-areas, being situated very eccentrical, either in absolute or in relative sense. (Tip of the tail, tips of the ears, ventral median line, frontal median line of the head, toes; they are all specimina of eccentrical defect-contrasts).

b. with definite non-pigmented skin-segments (phenomena of segmental-omission, segmental defect variability, segmental defect-contrasts).

5. Eimer's type of the transversally striped animals ought to be divided into two sub-divisions:

a. animals with broad, dark transversal stripes, which are less numerous than the segments of the body (fishes, sauria's, serpents). These broad transversal stripes correspond probably with groups of strongly pigmented segments, alternating with other groups that are

less pigmented. (A transversal serially ranged segmental excedent contrast).

b. animals with narrow dark transversal stripes, more numerous than the segments of the body (mammalia, e.g. zebra's). These stripes correspond with zones of intersegmental summation. (A transversal serially ranged intersegmental excedent contrast).

6. EIMER's type of the animals with longitudinal stripes includes:

a. fishes, in which the dark longitudinal stripes, or else the dark dots and spots ranged in long rows, correspond apparently with the points of entrance into the hypodermis of the skin-branches of the peripherical nerves. (An excedent contrast *ex introitu*).

b. amphibians and reptiles. Probably the precedent hypothesis holds likewise for these.

c. mammalia. In the viverridae the longitudinal stripes apparently have been produced by the confluence of rows of spots, which were originally distributed intersegmentally. (Pseudo-longitudinal stripes).

7. EIMER's spotted type in the mammalia includes:

a. Irregular spotting. This is caused by segmental excedent and defect-variability.

b. Uniform dotting. We may imagine this to have been produced by the fragmenting of stripes, that occur un-interrupted in kindred species of animals (leopards).

Meteorology. — "*On frequency curves of meteorological elements.*" Bij Dr. J. P. VAN DER STOK.

1. The application of the theory of probability to the results of meteorological investigations has hitherto been more limited than the nature of the data would lead us to expect.

It is not difficult to indicate the reason for this fact. Nearly all applications of the theory of errors to physical and astronomical problems are induced by the desire to determine a quantity with the greatest attainable precision; the remaining uncertainty affords a criterion for the value of the different methods employed and leads to experimental improvements, by means of which the errors, or departures from the average value, may be minimized.

These reasons for the application of the theory of errors fail in meteorology: for the greater part of meteorological quantities and climatological regions it is impossible to calculate average values within a reasonable time and with a moderate degree of precision and, if this were at all possible (e.g. for tropical stations), an increase

of precision would scarcely afford any advantage as we are unable to reduce the deviations by improving the observations. Moreover the knowledge of the most probable value is of minor importance as the frequency curves in general are very flat and we cannot attach the common idea of errors to the deviations which, after all, are more characteristic of meteorological conditions than absolute values.

Meteorological constants in the true sense of the word and to which the methods and terminology of the theory of errors are applicable, are nearly exclusively FOURIER constants, obtained by the analysis of periodical phenomena such as daily and annual variations, and to these it is certainly desirable to apply the criterion of the theory of errors more extensively than has hitherto been the case: the theory of errors in a plane can be immediately and advantageously used to get a clear understanding of the value of the results obtained.

If however we abandon this basis of the theory of errors and proceed upon the lines which have of late been followed by the sociological and biological sciences, the matter appears in a different light; in these sciences the principal object to be obtained is not so much the mean value as the occurrence of deviations, or rather the nature of the frequency curves.

Monthly means e.g. of barometric heights may be identical for January and July as far as the absolute values are concerned, but we may confidently expect the frequency curves for these months to bear a totally different character. It is also extremely probable that the frequency curves will show a considerable difference for places in different latitudes or differently situated in relation to the main tracks of depressions.

The constants which occur in the analytical expressions for these curves may then be considered as characteristics of the climate and, as in meteorology we possess more data than in most other branches of science, a more thorough study of details is possible.

The principal questions are:

a. In how far are monthly means in accordance with the common law of probability.

b. What is the form of the frequency curves constructed from daily means or from observations made at fixed hours in as far as these curves may be considered symmetrical.

c. An investigation of the skewness of these curves.

In this communication only the first of these problems will be considered.

2. The material chosen for this inquiry consists of:

1st. monthly means of barometric pressure at Helder, calculated for the 60 years period Aug. 1843 to July 1902, the total number being 720.

2nd. monthly means of barometric pressure at Batavia for 37 years, 1866—1902, altogether 444 data.

3rd. monthly means of atmospheric temperature for the whole of France during the 50 years period 1851—1900, altogether 600 data.

Up to 1873 the data for Helder have been taken from a meteorological journal kept by Mr. VAN DER STERR and, after his death, from the annals issued by the K. Met. Instituut.

A NEWMAN standard-barometer at Helder, which is known to have been in use as early as 1851, has recently been tested and does not show any appreciable errors, so that it may safely be assumed that also the records of the station-barometer are sufficiently accurate for our purpose.

The monthly means for Batavia have been taken from the returns published by the K. Magn. en Met. Observ., and those for France from ANGOT's *"Études sur le climat de la France, Température,"* published in the Ann. du Bureau Central Météor. de France, Année, 1900, I. Mémoires, Paris, 1902, p. 34—118.

Table I gives the results of the calculations for Helder.

Let ε be the deviations of the individual data from the corresponding general average value and n the number of data available, then:

$$M = \sqrt{\frac{|\varepsilon^2|}{n-1}}, \text{ mean deviation,}$$

$$\vartheta = \frac{|\varepsilon|}{n}, \text{ average deviation,}$$

$$h = \frac{1}{M\sqrt{2}}, \text{ factor of steadiness,}$$

$$h' = \frac{1}{\vartheta\sqrt{\pi}}, \text{ idem.}$$

$A =$ number of years required to obtain a general mean value with a probable error of $\pm$ 0.1 mm. for the barometric height and of $\pm$ 0°.1 C. for the atmospheric temperature.

This number, calculated from the formula:

$$\sqrt{A} = \frac{0.6745\ M}{0.1},$$

is given instead of the probable error of the result with a view of showing how difficult, if not how impossible, it is to fix normal

values of meteorological elements, at least in high latitudes. The application of this formula is justified by the consideration that monthly means for a given month may, as far as our actual knowledge goes, be regarded as independent of each other, whereas e.g. daily means are certainly not so.

If the deviations are distributed according to the normal, exponential law:

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the quantity h' must be equal to h. Another criterion to ascertain whether the distribution of deviations is regulated by the normal law, as advocated by CORNU [1]), is obtained by calculating π by means of the formula:

$$\pi = \frac{2\,M^2}{\vartheta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

it is equivalent to the criterion previously mentioned as it holds only when $h = h'$.

The quantities M and h may be regarded as a measure of the

TABLE I. Monthly means of barometric height, Helder.

	M	ϑ	h	h'	A	π
January.....	5.01 mm.	4.04 mm.	0.141	0.140	1149	3.083
February.....	4.85	3.82	0.146	0.148	1071	3.222
March.....	4.21	3.45	0.168	0.164	805	2.971
April.....	3.36	2.74	0.211	0.206	514	3.007
May.....	2.34	1.91	0.302	0.296	250	3.022
June.....	2.22	1.76	0.318	0.321	225	3.190
July.....	2.09	1.65	0.339	0.343	198	3.212
August	2.12	1.69	0.334	0.334	206	3.158
September.....	3.01	2.45	0.235	0.230	411	3.079
October	3.35	2.62	0.211	0.215	510	3.262
November.....	3.74	3.02	0.189	0.187	637	3.063
December.....	4.99	4.00	0.142	0.141	1132	3.106
Mean.....						3.198

1) Ann. de l'Observ. de Paris. XIII, 1876.

variability and steadiness of the climate from year to year in so far as this is determined by the oscillation of the atmospheric pressure. By analogy to the secular variation of the elements of terrestrial magnetism this instability might also be called secular variability.

Assuming this criterion to be correct, it appears from Table I that there is every reason to suppose that at Helder the deviations follow the normal law, the average value of π not differing more than 1.8 % from the real value.

On comparing the climate at Helder, which is highly variable from year to year, with the climate at Batavia (in so far as in this case also the variability of atmospheric pressure may be taken as a measure), we find totally different conditions.

A period of about ten years for the Eastmonsoon, and of twenty years for the Westmonsoon months is already sufficient to obtain total monthly means of the barometric height with a probable error of $\pm$ 0.1 mm. and for the dry months the available series of 37 years is quite sufficient to obtain a degree of certitude twice as great.

TABLE II. Monthly means of barometric height, Batavia.

	M	ϑ	h	h'	A	π
January.....	0.84 mm.	0.71 mm.	0 845	0.792	32	2.759
February.....	0.75	0.62	0.938	0.917	26	3.004
March.....	0.63	0.52	1.115	1.085	18	2.974
April.....	0.42	0.36	1.701	1.581	8	2.715
May.....	0.44	0.32	1.603	1.751	9	3.752
June.....	0.40	0.28	1.779	1.990	7	3.931
July.....	0.44	0.34	1.604	1.610	9	3.282
August.....	0.47	0.33	1.492	1.689	10	4.028
September.....	0.44	0.35	1.598	1.606	9	3.173
October.....	0.51	0.41	1 375	1.370	12	3.118
November.....	0.65	0.53	1.088	1.063	19	2.999
December.....	0.61	0.49	1.166	1.155	17	3.086
Mean.....						3.235

The application of the criterion as to whether the deviations follow

the normal law leads to a far less satisfactory result for this place than for Helder. The two values h and h' of the factor of steadiness show considerable and systematic discrepancies, the calculated values of π for May to August being collectively too great, and those for the other months too small. Although the total mean, 3.235, does not differ more than 3 % from the real value, these differences amount to + 15.7 % in the five dry months and to — 6.5 % in the seven months of the wet season.

Here, therefore, the secular variability cannot be regarded as a purely accidental quantity unless another law, more complicated than the normal one, applies and which is in some degree dependent upon the monsoons. This might be the case if the atmospheric pressure were dependent (and in a different manner in different seasons) upon another factor, for instance the temperature, the variability of which might still be according to the law of accidental quantities.

Similar systematic differences, varying with the season, between the calculated and the real value of π are not apparent in the results of the calculations for the atmospheric temperature in France, and the general average value of π does not differ from the real value more than 0.13 %.

TABLE III. Monthly means of atmospheric temperature, France.

	M	ϑ	h	h'	A	π
	°	°				
January.....	2.07 C.	1.73 C.	0.341	0.326	195	2.869
February.....	2.03	1.70	0.356	0.332	188	2.855
March.....	1.59	1.25	0.446	0.452	115	3.230
April.....	1.20	0.92	0.588	0.616	66	3.444
May.....	1.32	1.07	0.536	0.529	79	3.067
June.....	1.14	0.91	0.629	0.623	59	3.150
July.....	1.29	1.00	0.548	0.565	76	3.347
August.....	1.08	0.88	0.653	0.641	53	3.029
September.....	1.19	0.94	0.594	0.600	64	3.205
October.....	1.25	1.02	0.565	0.551	71	2.991
November.....	1.50	1.22	0.472	0.464	102	3.043
December.....	2.41	1.84	0.294	0.306	264	3.418
Mean.....						3.137

In the paper already quoted, Mr. ANGOT assumed that the deviations do not show systematic differences in different months, and he subjects the deviations taken conjointly to the criterion of the law of errors.

This assumption is not justified by the results given in Table III, from which it is evident that the values of h are subject to considerable and systematic variations and, if a satisfactory agreement is still found between theory and observation, this can only be accounted for by the fact that the probability of the occurrence of deviations between fixed limits is expressed in a number of decimals too restricted to indicate the differences which, as for Helder and Batavia, must here exist between theory and practice.

No more can it be affirmed that, if a satisfactory accordance exists between the calculated and the observed number of deviations between given limits, the average value will also be the most probable one. In applying this criterion, as well as in calculating h' and π, a possible (and probable) skewness of the frequency curve is not taken into account because, by treating the deviations without regard to their sign, symmetry with respect to the ordinate of the centre of gravity of the figure is tacitly assumed.

As the number of years over which the observations extend is still far too small to allow frequency curves to be drawn for each month separately, it is still worth while to consider the deviations collectively, provided that at the same time the question be put, what form the law of deviations will assume when they are composed of groups which individually follow the normal law, the factor of steadiness being different for different groups. Even then the available data are insufficient to indicate with certainty a small degree of skewness in the frequency curve, so that only the symmetrical form can be sought for.

3. If, as in our case, the different groups occur with equal (sub) frequency, it is not difficult to indicate in what respects such a curve, the resultant of many elements, must differ from the normal curve. The groups characterised by large factors of steadiness will raise the number of small deviations above the number corresponding with an average factor and contribute only in a small degree to the number of large deviations, whereas, on the contrary, flat curves with small factors will give rise to a greater number of large deviations than is consistent with the normal law. Deviations of average magnitude will then occur to a less degree than is required by the common law; consequently in drawing the two curves, they will be seen to intersect at four points, as a minimum.

In a paper [1]) published some years ago, SCHOLS has drawn the attention to the fact that differences of this description are almost always found when sufficiently extensive series of errors are put to the test of the normal law; in this paper he shows that these differences cannot be explained by the omission of terms in BESSEL's development of the exponential law and suggests that their origin must be sought for in the superposition of observations of different degrees of precision.

In the observations alluded to by SCHOLS, it will in general not be possible to estimate these degrees of precision any more than the relative subfrequencies with which the different groups are represented in the result; in the case of monthly means such as are being discussed here, the factors of steadiness are approximately known and the subfrequencies of the different groups are all identical.

If we arrange the 12 groups according to increasing values of h, it appears that we may take its change to be uniform; consequently it is possible to find an approximate solution of the problem in finite form.

We have then to consider h as a variable quantity z and to ask what form the expression will assume for a sum of elementary surfaces:

$$C\int_{-\infty}^{\infty} e^{-z^2x^2}\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

if z varies in a continuous manner from h to H. If the subfrequency of these elementary groups be also regarded as a function of z (which occurs e.g. in the case of wind-frequencies), (3) must be equated to $\varphi(z)\,dz$, $\varphi(z)$ being subject to the condition:

$$\int_h^H \varphi(z)\,dz = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The constant C is determined by the expression

$$C = \frac{z\varphi(z)dz}{\sqrt{\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and if, as in our case,

$$\varphi(z) = c$$

$$c = \frac{1}{H-h} \quad , \quad C = \frac{zdz}{(H-h)\sqrt{\pi}},$$

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[1]) Versl. Wis. Nat. Afd. K. Akad. Wet. I. 1893 (p. 194—202).

the resulting probability of a deviation being situated between x and $x + dx$ is then:

$$\frac{dx}{(H-h)\sqrt{\pi}} \int_h^H z\, e^{-z^2x^2}\, d\,z$$

and the equation of the frequency curve:

$$y = \frac{1}{2(H-h)\sqrt{\pi}} \left[\frac{e^{-h^2x^2} - e^{-H^2x^2}}{x^2} \right] \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Developing this expression we may put:

$$y = \frac{H+h}{2\sqrt{\pi}}\, e^{-\frac{H^2+h^2}{2}x^2} \left[1 + \frac{(H^2-h^2)^2}{2^2 \,.\, 3\,!}\, x^4 + \frac{(H^2-h^2)^4}{2^4 \,.\, 5\,!}\, x^8 \ldots \right] . \quad (7)$$

If we put:

$$\mu_n = 2 \int_0^\infty x^n y\, dx$$

we find with the help of:

$$2 \int_0^\infty \tau^n\, e^{-\tau^2}\, d\tau = \Gamma\left(\frac{n+1}{2}\right)$$

and

$$\int_0^\infty \frac{e^{-pz} - e^{-qz}}{z}\, dz = log\, \frac{q}{p},$$

for the moments of different order with respect to the maximum ordinate:

$$\mu_0 = 1 \quad , \quad \mu_2 = M^2 = \frac{1}{2Hh}$$

$$\mu_1 = \vartheta = \frac{1}{(H-h)\sqrt{\pi}}\, log\, \frac{H}{h} \quad , \quad \mu_3 = \frac{1}{2\sqrt{\pi}}\, \frac{H+h}{H^2h^2} \quad . \quad . \quad (8)$$

From a series of deviations following the law (6) the two characteristic constants H and h can be derived by computing the moments of the second and third order. They are found to be equal to the roots of the quadratic:

$$X^2 - p\,X + q = 0$$

$$p = \frac{\mu_3\sqrt{\pi}}{2{\mu_2}^2} \quad , \quad q = \frac{1}{2\mu_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we had put a similar series to the test of the normal law (1) we should have found for the equation of the frequency curve;

$$y = \sqrt{\frac{Hh}{\pi}}\, e^{-Hhx^2}$$

or

$$y = \sqrt{\frac{Hh}{\pi}}\, e^{-\frac{H^2+h^2}{2}x^2}\left[1 + \frac{(H-h)^2}{2.1!}x^2 + \frac{(H-h)^4}{2^2\,2!}x^4 \ldots\right]. \quad (10)$$

On comparing this expression with (7) it is at once seen that in this manner too great a number of small deviations must be found, as the module of the deviation zero, computated by (10)

$$\sqrt{\frac{Hh}{\pi}},$$

is always smaller than that derived from (7):

$$\frac{H+h}{2\sqrt{\pi}}.$$

The position of the four points where the two curves intersect are found by equating the expressions (7) and (10); if the development can be stopped at the third term they are given by the roots of the biquadratic:

$$p\,X^4 - q\,X^2 + s = 0 \ldots\ldots\ldots (11)$$

$$p = \frac{(H+h)^2}{6} - \sqrt{Hh}\,(H-h)^2 \quad , \quad q = 4\sqrt{Hh}$$

$$s = \frac{4\,(\sqrt{H}-\sqrt{h})^2}{(H-h)^2}.$$

With the help of the form. (8) for ϑ, it can be shown that, if a series of figures follows the law (6) the computation of π according to (2) must necessarily lead to values which are somewhat too high:

$$\frac{2\mu_2}{\vartheta^2} = \pi\,\frac{(H-h)^2}{Hh}\left(log\,\frac{H}{h}\right)^{-2}$$

Putting:

$$H + h = p\,, \quad H - h = q\,,$$

we find:

$$log\,\frac{H}{h} = 2\,\frac{q}{p}\left(1 + \frac{1}{3}\,\frac{q^2}{p^2} + \frac{1}{5}\,\frac{q^4}{p^4} + \cdots\right)$$

$$\frac{2\mu_2}{\vartheta^2} = \pi\,\frac{1 + \frac{q^2}{p^2} + \frac{q^4}{p^4} + \cdots}{\left(1 + \frac{1}{3}\,\frac{q^2}{p^2} + \frac{1}{5}\,\frac{q^4}{p^4} + \cdots\right)^2} > \pi \quad \ldots (12)$$

4. In the following applications of these reasonings to deviations taken collectively for all months, the frequencies are reduced to a total number of 1000: by exponential law is understood the simple, normal law of errors (1).

TABLE IV. Barometer, Helder.

Dev. mm.	Observ.	Exp. L	Diff.	Dev. mm.	Observ.	Exp. L.	Diff.
0.0—0.45	104	100	+ 4	5.95—6.45	21	25	— 4
0.45—0.95	129	108	+21	6.45—6.95	17	19	— 2
0.95—1.45	121	106	+15	6.95—7.45	14	15	— 1
1.45—1.95	101	100	+ 1	7.45—7.95	7	11	— 4
1.95—2.45	97	92	+ 5	7.95—8.45	18	8	+10
2.45—2.95	86	84	+ 2	8.45—8.95	8	6	+ 2
2.95—3.45	68	75	— 7	8.95—9.45	10	4	+ 6
3.45—3.95	50	65	—15	9.45—9.95	7	3	+ 4
3.95—4.45	43	56	—13	9.95—10.45	2	2	0
4.45—4.95	38	47	— 9	10.45—10.95	1	1	0
4.95—5 45	31	39	— 8	10.95—11.45	0	0	0
5.45—5.95	25	32	— 7	11.45—11.95	2	2	0

$$\mu_1 = 2.769\ ,\ \mu_2 = 12.867\ ,\ \mu_3 = 77.427\ ,$$

$$h\,(Exp.\ L) = \frac{1}{\sqrt{2\mu_2}} = \sqrt{Hh} = 0.1971,$$

π (form. 2) $= 1.069 \times 3.142$.

Points of intersection observ. near dev. 2.95 and 7.95,

$$H = 0.2712\ ,\ h = 0.1433 \text{ (form. 9)}.$$

π (form 12) $= 1.044 \times 3.142$.

Points of intersection (form. 11) at dev. 2.60 and 9.19.

The sums of the differences between the limits of the observed points of intersection are, as given in Table IV, +48, —70, +22.

If we also wish to compare these quantities with the result of the theory, we have to integrate (6) between the limits deduced from (11). For the limits α and zero we find the frequency :

$$\frac{1}{\alpha(H-h)\sqrt{\pi}}\left[e^{-\alpha^2H^2}-e^{-\alpha^2h^2}\right]+\frac{2}{(H-h)\sqrt{\pi}}\left[H\int_0^{\alpha H}e^{-\tau^2}d\tau-h\int_0^{\alpha h}e^{-\tau^2}d\tau\right] \quad (13)$$

By means of this formula we find between the limits calculated by means of (11):

α	Form. (6)	Form. (10)	Diff.	Obs.
0—2.60	553	509	+44	+48
2.60—9.19	440	476	—36	—70
9.19—etc.	7	15	— 8	+22

As the situation of the second point of intersection according to the observations (7.95) shows a rather large discrepancy with that given by theory (9.19), it is natural that only the sums of the positive differences between the limits zero and the first point of intersection agree closely.

Taken as a whole it may be stated that the secular variability of barometric pressure at Helder is regulated by the law of accidental events as completely as might have been expected considering the scantiness of the material available.

A possible skewness of the curve is left out of consideration as has been already remarked; it can, however, be but unimportant as in 720 deviations 364 are positive and 356 negative.

The same cannot be ascertained of the secular variability of barometric pressure at Batavia; the differences between the observed frequencies and those calculated according to the exponential law are not of such a well marked description as for Helder, so that a determination of the points of intersection is out of the question; their situation can only be calculated as a result of theory.

TABLE V. Barometer, Batavia.

Dev. mm.	Observ.	Exp. L.	Diff.	Dev. mm.	Observ.	Exp. L.	Diff.
0.000—0.095	146	135	+11	0.995—1.095	34	25	+ 9
0.095—0.195	149	136	+13	1.095—1.195	7	18	—11
0.195—0.295	126	129	— 3	1.195—1.295	7	12	— 5
0.295—0 395	117	118	— 1	1.295—1.395	10	8	+ 2
0.395—0 495	101	104	— 3	1.395—1.495	5	5	0
0.495—0.595	95	89	+ 6	1.495-1.595	7	3	+ 4
0 595—0.695	50	74	—24	1.595-1.695	2	2	0
0.695—0 795	52	59	— 7	1 695—1.795	2	1	+ 1
0.795—0.895	59	46	+13	1.795—1.895	0	1	— 1
0.895—0.995	29	35	— 6	1.895—etc.	2	0	— 2

$$\mu_1 = 0.4395,\ \mu_2 = 0.3156,\ \mu_3 = 0.2915.$$

$$h\ (\text{Exp. L.}) = 1.2586,\ \pi\ (\text{form. (2)} = 1.040 \times 3.142,$$

$$H = 1.989,\ h = 0.796.$$

$$\pi\ (\text{form. 12)}) = 1.0115 \times 3.142.$$

Points of intersection (form 11) at dev. 0.399 and 1.620. For the sums of deviations between these limits we find (form 13):

α	Form. (6)	Form. (10)	Diff.	Obs.
0 — 0.399	559	522	+ 37	+ 17
0.399 — 1.620	431	474	— 43	— 19
1.620 — etc.	10	4	+ 6	+ 2

It appears from these results that the calculation of π cannot always be regarded as a good criterion of the variability being regulated by the law of accidental events. From a series of numbers, composed, as the barometric departures for Batavia are, of groups which follow neither the simple normal law nor the more complicated law (6), still the calculation of π leads to a value which is correct within 1%.

TABLE VI. Temperature, France.

Dev. C°.	Observ.	Exp L.	Diff.	Dev. C°.	Oserv.	Exp. L.	Diff.
0.00—0.15	73	78	— 5	2.15—2.35	27	36	— 9
0.15—0.35	113	101	+12	2.35—2.55	18	29	—11
0.35—0.55	108	98	+10	2.55—2.75	8	24	—16
0.55—0.75	87	95	— 8	2.75—2.95	25	19	+ 6
0.75—0.95	100	88	+12	2.95—3.15	15	15	0
0.95—1.15	83	82	+ 1	3.15—3 35	12	11	+ 1
1.15—1.35	77	74	+ 3	3.35—3.55	8	9	— 1
1.35—1.55	60	66	— 6	3.55—3.75	12	6	+ 6
1.55—1.75	70	59	+11	3.75—3.95	5	5	0
1.75—1.95	58	50	+ 8	3.95—4.15	2	3	— 1
1.95—2.15	30	43	—13	4.15—etc.	9	9	0

$$\mu_1 = 1.207,\ \mu_2 = 2.394,\ \mu_3 = 6.703.$$
$$h\,(\text{Exp. L.}) = 0.4570,\ \pi\,(\text{form. (2)}) = 1.046 \times 3.142,$$
$$H = 0.7627,\ h = 0.2739,$$
$$\pi\,(\text{form. 12}) = 0.140 \times 3.142.$$

Position of points of intersection (form. (11) at dev. 1.09 and 4.87. Sums of deviations between these limits (form. 13):

α	Form. (6)	Form. (10)	Diff.	Obs.
0 — 1.09	565	519	+ 46	+ 22
1.09 — 4.87	429	480	— 51	— 22

As a general result of this investigation it can be stated that, according to theory, in all three series the number of small deviations is greater than the simple exponential law would require, but to a somewhat less degree than would follow from the law formulated in (6).

The deviations of barometric pressure at Helder are in almost perfect accordance with this frequency law and, therefore, for each month separately with the normal law; the curve of deviations of atmospheric temperature in France still shows many irregularities, but, in general, it accords well with the law of form. (6); the secular variability of atmospheric pressure at Batavia is not regulated by the law of accidental events and its frequency curve shows characteristic peculiarities in different seasons.

Microbiology. — *"Methan as carbon-food and source of energy for bacteria"*. By N. L. Söhngen. (Communicated by Prof. M. W. Beijerinck).

Methan, which is incessantly produced from cellulose in the waters and the soil, through the agency of microbes, and which, since vegetable life became possible on our planet must have been formed in prodigious quantities, yet occurs only in traces in our atmosphere.

As this gas is very resistant against chemical influences its disappearance in this way is highly improbable. But the conversion of methan into carbon dioxid and water produces a considerable quantity of heat, and so it seemed worth investigating whether there should exist any organic beings capable of feeding and living on it.

In the first place green plants were examined as to their power of decomposing methan in the light. To this end some waterplants were chosen, which seemed to offer most chance of success, considering that the formation of methan, as an anærobic process, takes especially place in stagnant waters.

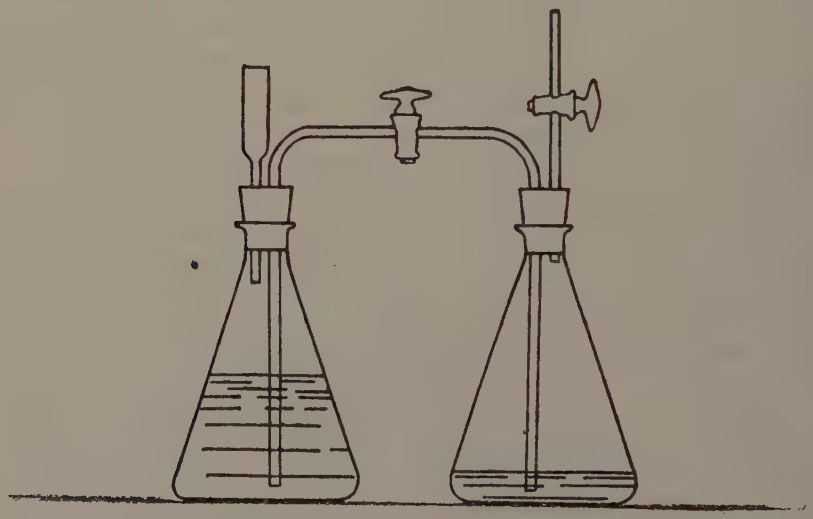
In this way positive results were obtained with several species of plants as *Callitriche stagnalis, Potamogeton, Elodea canadensis, Batrachium, Hottonia palustris, Spirogyra*. So, for example, in one of the experiments in the light of a window to the North, with *Hottonïo palustris*, put in a flask containing 500 cc. of methan and 500 cc. of oxygen, and inversely placed in a vessel filled with water, all the methan disappeared from 7—21 May, so within a fortnight.

In the dark, also, absorption of methan was with certainty observed.

However, the lapse of time preceding the first perceptibility of the process in different experiments with the same species of plant, varied very much, but when once set in it went on rapidly. When it was moreover observed, that by carefully washing the plants the setting in of the absorption was much slackened, whilst it seemed probable that just then an acceleration would follow in case the plant itself absorbed the methan, and especially when furthermore the absorption was observed to take place only after a slimy film had covered the water in the flask, it became evident that the oxidation was not caused by the green plant itself, but by microbes living on it surface.

In order to study the process more exactly an apparatus was constructed allowing us to pursue the absorption as well qualitatively as quantitatively.

It consists, as shown in the figure, of two ERLENMEYER-flasks of ± 300 cc., each closed by an indiarubber stopper with two perforations and joined by a twice curved glass tube reaching to the bottom of the flasks, which bears in the middle a glass cock. The flask, destined for the cultivation of the bacteria, bears, in the second perforation of the stopper, a tube with a glass cock to admit the gasmixture; the other flask is fitted with a glass tube filled with cotton wool.



The use of this apparatus is as follows: For the crude culture the first mentioned flask is quite filled with the culture liquid:

Destilled water	100
$K^2 HPO^4$	0,05
$NH^4 Cl$	0,1
$Mg NH^4 PO^4$	0,05
$Ca SO^4$	0,01

and inoculated with garden soil, sewage or canalwater, of which the two last cause the quickest growth.

By the cock on the first flask a measured quantity of oxygen and methan is admitted by means of a gas burette. The liquid is thereby pressed into the other flask, and when it has lowered until a layer of about 1 cM. remains in the first flask, then the middle-cock is shut and at last the admission-cock.

The cultivation is effected at about 30° C. After a period, varying from 2—4 days, a film is observed on the liquid, which rapidly increases in thickness and then shows a distinct pink colour. Beneath the film the liquid, clear at first, begins te display a considerable turbidity caused by foreign microbes, which feed on the dead bacterial bodies of the floating film. Later on a great number of amoebes and monads develop in the film and in the liquid, evidently at the expense of the methan bacteria, no other material for food being present. In the other flask no film appears on the liquid.

Transports to a same liquid in an apparatus like the former, easily produce a new film, and, when garden soil is used for the infection, it grows even faster than in the crude culture.

An analysis of the gas after about a week, shows that the methan has quite or partly disappeared whilst a considerable quantity of carbonic acid is formed. The film is found chiefly to consist of bacteria belonging to one single species, which has proved to be the microbe which makes the methan disappear. It is a short, rather thick rodlet, immobile in the film, mobile or immobile in the plate cultures. Always the individuals are united by a layer of slime.

The length of this bacterium, which will provisorily be called *Bacillus methanicus,* is 4—5 μ, its thickness 2—3 μ.

It is not yet ascertained whether this species has already been found under other conditions of life and described elsewhere without the knowledge of its relation to methan. The question whether there exists only one or more than one species possessing the faculty to live on methan is also subjected to further investigation.

The methan bacterium is easily obtained in pure culture by cultivation on washed agar, containing the necessary salts, at a temperature of circa 30° C., in an atmosphere of $^1/_3$ methan and $^2/_3$ air,

with which an exsiccator is filled and into which the plates are introduced.

By streaking a young film from a liquid culture on the said solid medium already on the second day nearly pure slightly turbid colonies are obtained, quite distinguishable by their size and their slimy and lightly pink-coloured appearance. Such a colony, when early inoculated into the above apparatus forms, after some days, another bacterial film.

The methan, being in all the experiments the only source of carbon, necessarily at the same time must serve as food and as source of energy.

The quantity of carbonic acid in the culture flask indicates the amount of methan which has served as source of energy. The quantity of methan used for the formation of the bacterial bodies may be measured by subtracting the quantity of produced carbonic acid, expressed in cc., from the volume of disappeared methan.

So for example it was found that in an experiment in which were added successively 225 cc. CH^4 and 320.7 cc. O^2 to 102 cc. of liquid, the flasks contained after a fortnight

78 cc. CO^2
no CH^4
172 cc. O^2.

In the culture liquid 21 cc. of carbonic acid were solved, so that 126 cc. of methan had been assimilated for building up the bacterial bodies, and 78 + 21 cc. CH^4 for the respiration, 148.7 cc. of oxygen being assimilated.

Another experiment gave the following result.

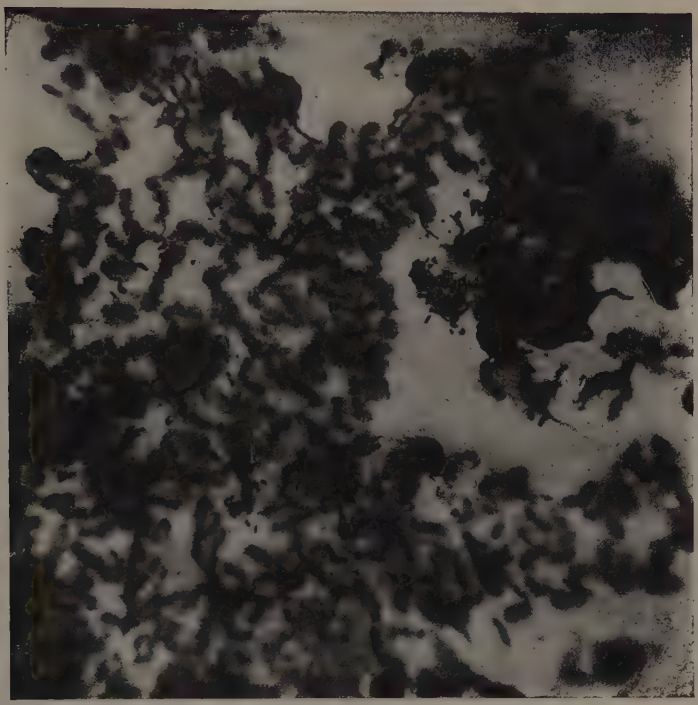
Successively added 200 cc. CH^4
and 331 cc. O^2.
to 108.5 cc. liquid.

After two weeks the gas contained

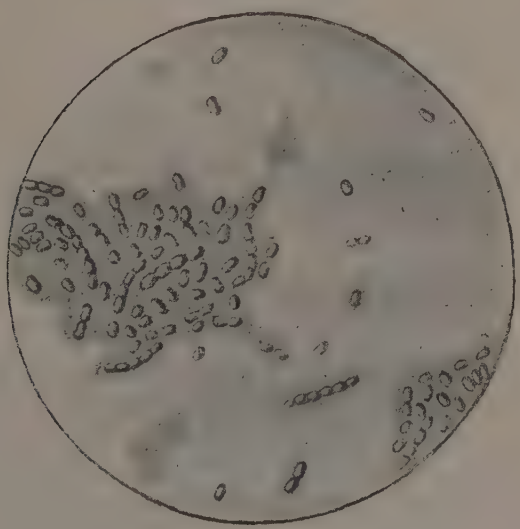
72.8 cc. CO
39 cc. CH^4
138 cc. O^2.

In the culture liquid 18 cc. of carbonic acid were solved. Hence, 73.2 cc. CH^4 had been assimilated for the formation of the bacterial bodies, whilst 90.8 cc. CH^4 were converted into CO^2.

Some oxidation experiments were performed with permanganate and sulphuric acid, in order to prove that a large quantity of organic material had accumulated. Thus, 100 cc. of the culture liquid, described in the first experiment, consumed:



Bacillus methanicus (800).
Crude film on culture liquid in methan-oxygen atmosphere. Between the bacteria mucus occurs.



Bacillus methanicus (1000).
Pure culture on agar with salts in methan-oxygen atmosphere.

Before the cultivation 0 cc. $\frac{1}{10}$ normal $KMnO^4$.

After the cultivation 48.3 „ „ „

At a second experiment 100 cc. consumed:

Before the cultivation 0 cc. $\frac{1}{10}$ normal $KMnO^4$.

After the cultivation 26.5 „ „ „

Even this rough estimation gives the convincing result that much organic matter is formed from the methan. Hence it follows that methan is the starting point for the production of a relatively rich flora of microbes, which as said above, may even at an early period contain amoebes and monads living from the methan bacteria.

There can thus be no doubt but methan is, though indirectly, of importance as a fish-food in the waters, as the said flora certainly serves as such.

Further investigations concerning the natural history of the methan bacteria and the relation between the assimilated methan and the amount of organic matter produced are in execution.

H. Kaserer (Zeitschrift für das Versuchswesen in Oesterreich, Bd. 8 p. 789, 1905) seems also to have observed bacteria living on methan, but he gives no particulars.

Microbiologic Laboratory
of the Technic High School at Delft.

Physics. — "*Determination of the* Thomson-*effect in mercury.*" By C. Schoute. (Communicated by Prof. H. Haga.)

This determination has been executed as a sequel to that, undertaken by Prof. H. Haga, and published in the "Annales de l'Ecole Polytechnique de Delft, I, 1885, p. 145; III, 1887, p. 43."

A detailed account of the way, in which the experiments were carried out, has been given in my "Dissertation". The results mentioned here were partly obtained afterwards.

The value of the Thomson-constant was expressed by a relation, got by integration of the differential equation, which Verdet has given for the points of an unequally heated homogeneous conductor, when an electric current passes through it.

If the distribution of temperature is considered, after it has grown constant, and in some portion of the conductor, confined by two parts of a constant temperature, this equation is integrable, and the integral is quite simple for the points halfway between these limits of constant

temperature, when all over the part between them the external exchange of heat, by conduction, convection and radiation is small enough to be disregarded with respect to the other thermal effects.

The THOMSON-constant σ may then be expressed:

$$\sigma = \frac{i\,w}{2Jq\,\frac{U}{l}} \cdot \frac{\Delta_{Th}u}{\Delta_{J}u},$$

wherein i represents the strength of the current; w the resistance; J the mechanical equivalent of heat; q the section of the conductor; U the difference of temperature between the two parts of constant temperature; l the distance between those two parts; $2\,\Delta_{Th}u$ the change of temperature which manifests itself in the middle-section when the current is reversed; and $\Delta_{J}u$ the rise of temperature in the same section according to Joule's law.

In order to be able to measure $4\Delta_{Th}u$ instead of $2\Delta_{Th}u$ the mercury was investigated in a U-shaped glass tube, put in a vertical position, the curved part up. The upper part of this U-tube was enclosed in a glass bulb, in which different fluids (acetone, water, aniline, glycerin) could be kept boiling by an electric current. In this way the upper part was kept at a constant temperature. For the same purpose the bottom parts of the legs of the U-tube, which were closed by small rods of platinum, were placed in running tapwater.

In the parts of non-uniform temperature this temperature was measured in sections halfway between the constant limits. If, after the current has been sent through in one direction, there should exist a certain difference of temperature between the two middle-sections, this difference will suffer a change of $4\Delta_{Th}u$ by reversing the current, if the condition about the external exchange of heat is fulfilled.

Therefore the parts of non-uniform temperature were enclosed in a large vacuum-tube, for the greater part of glass, with a brass bottom and, for the sake of practical advantages, the glass boiling bulb and part of the condenser upon it were also enclosed in this tube.

In order to measure $\Delta_{J}u$, separate experiments were made, with as nearly as possible the same current. By making the current go first through one leg and then through the other the difference in temperature of the middle-sections was varied by $2\,\Delta_{J}u$.

For measuring the temperature in the mercury the thermo-electric difference between this metal and platinum was used. Different kinds

of platinum acted quite differently in this regard. The strongest thermo-currents were obtained with *Pt Ir* of 10 to 20 %. A wire of this platinum was fused into each of the legs of the *U*-tube, as accurately as possible in the middle-section. These wires being connected and a sensitive galvanometer being introduced into the circuit, the temperature-differences Δ_{Thu} and Δ_{Ju} could be measured in proportion. Should we have wished to measure each of those quantities separately, it would have been necessary to determine the thermo-electric constants of this platinum with regard to mercury.

The unequality in temperature in the middle wires caused by an inevitable lack of symmetry in the *U*-tube was compensated by means of another thermo-couple. After each series of observations the galvanometer deflection, given by this couple with a known resistance and a known difference in temperature between the points of contact, was measured, in order to eliminate changes in the sensibility of the galvanometer or in the distance of the scale.

The quotient $\frac{U}{l}$ was determined indirectly. If the external exchange of heat could be neglected, the temperature-gradient must be the same all over the parts of non-uniform temperature, so long as the current did not pass through the mercury, apart from the distribution of temperature near the limits. And in the middle-section the gradient of temperature would remain very approximately the same, when the current did pass through it. Therefore the quantity $\frac{U}{l}$ could be said to be equal to the temperature-gradient in the middle-sections.

To measure this gradient in each of the legs of the *U*-tube on both sides of the middle-section at a given short distance both above and below it, another wire of platinum was fused in. The temperature-difference between these sets of wires divided by their distances was put for $\frac{U}{l}$.

The wires last mentioned were of a kind of platinum of which the thermo-electrical constants with regard to mercury had been accurately determined beforehand. As the same thing cannot be said about the wires in the middle-sections it is impossible to say anything definite about the uniformity of the gradient resulting from the experiments as they have been made. Preparatory experiments however have shown, that when l does not exceed certain limits, the gradient is sufficiently uniform.

Much trouble has been caused by wild thermo-electric currents.

Especially in a commutator for the galvanometer-current these difficulties arose. Contacts made by solid homogeneous copper have given the greatest satisfaction. With this arrangement for measuring the temperature the current through the *U*-tube, the chief current, had to be cut off for a moment during the reading of the galvanometer. Therefore the galvanometer commutator was combined with an interrupter for the chief current.

Changes in the meridian during the experiments were eliminated by noting, before the deflection, the position of the galvanometer-mirror when at rest. This position was more or less affected by the magnetic field of the chief current, but this obstacle was overcome by systematically combining readings with reversed chief current and galvanometer-current.

The galvanometer, made by CARPENTIER, was of the THOMSON-type. Provided with a sensitive set of magnets after PASCHEN, suspended by a quartz-fibre of $\pm 7\,\mu$, with electromagnetical damping and with coils of small resistance (2,76 Ω), this instrument answered to all the special requirements of the problem.

The strength of the current was determined by measuring the drop of the potential at the ends of a known resistance, and comparing this with that at the poles of a WESTON-element. The potential differences were measured with a five-cell quadrant-electrometer (H. HAGA, These Proc. I p. 56).

The course of the experiments was the following:

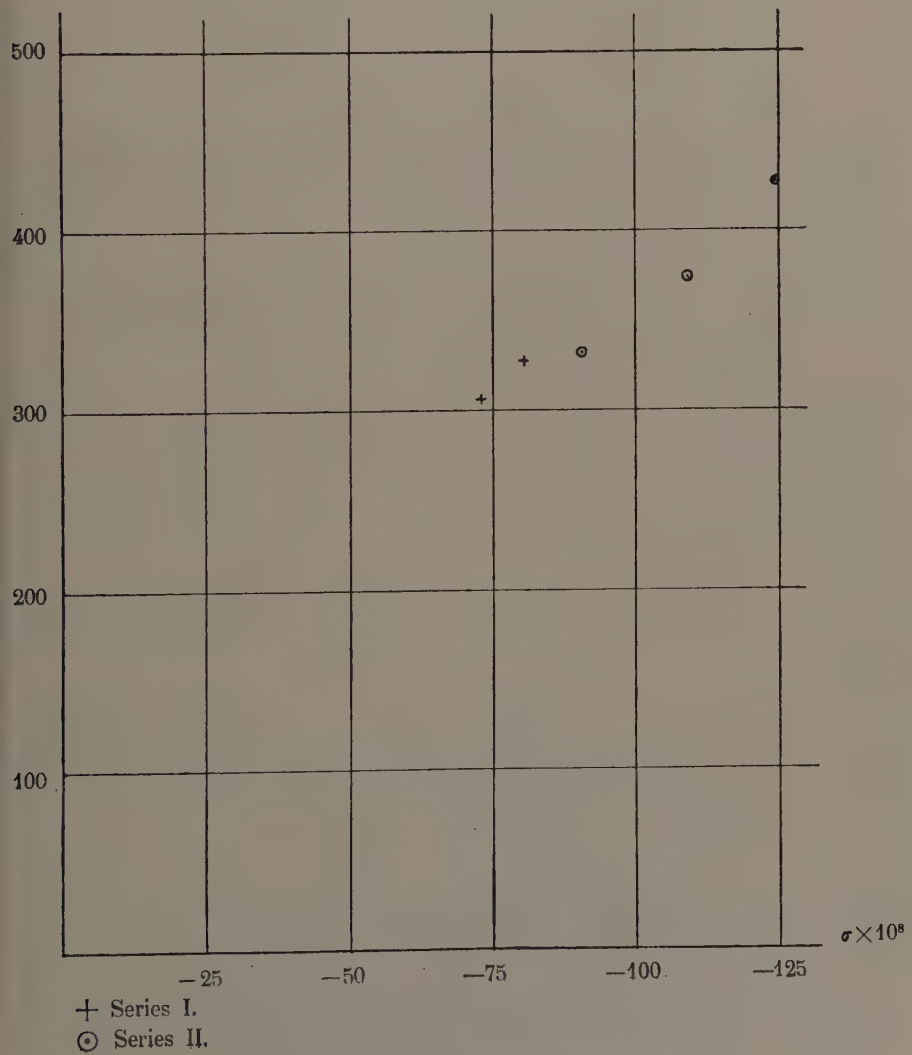
A sufficiently long time beforehand the fluid in the boiling-recipient was set boiling and the tapwater was allowed to run. Then the current in the *U*-tube was closed. When the distribution of the temperature had grown constant, the positions of the galvanometer resp. when at rest and deflected were read. After five minutes these readings were repeated, but now the commutator for the galvanometer was closed in the opposite direction. Then the current in the *U*-tube was reversed and after 10 or 15 minutes the galvanometer-readings were resumed. In a corresponding way the measuring of the Joule-heat was carried out.

In each series 8 deflections were read, as well for the determination of $\Delta_{Th}u$ as of $\Delta_{J}u$; first four of one quantity, then eight of the other, and again four of the first. In the meanwhile during the time necessary for the temperature to become constant, the current strength was measured from time to time, and the temperature of the running water was read. In this way the following results have been obtained:

	t	$\sigma \times 10^8$
I	32°	— 73
	53°	— 80
II	58°	— 90
	100°	— 108[5]
	154°	— 124

The values I are averages of the results of four series each, which have been given in my "Dissertation".

The values II have been obtained with another similar instrument



under about the same conditions. They represent the averages of resp. 2, 2 and 1 series.

The meaning of those values for σ is: When a current of one ampère passes through a column of mercury, the THOMSON-effect will cause a quantity of heat, equal to σ (expressed in gram-calories) to be developed in one second between two consecutive sections of the temperatures $t - \frac{1}{2}^\circ$ and $t + \frac{1}{2}^\circ$, if the current goes in the direction of the increasing temperatures.

As the diagram added shows, the values I and II for σ lie all but in straight lines, passing through the origin, which means, that the THOMSON-effect is proportional to the absolute temperature (T).

The values II give $\frac{\sigma}{T} = -284 \times 10^{-11}$, and the combination of I and II give $\frac{\sigma}{T} - 267 \times 10^{-11}$.

It is not clear what has caused the difference between I and II. May be it is the effect of some difference in purity of the mercury which is known by experiments on other substances to strongly affect the THOMSON-constant.

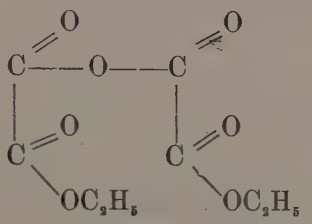
Chemistry. — Prof. FRANCHIMONT presents a communication from Dr. D. MOL on an investigation commenced in 1903 as to the *"ester anhydrides of dibasic acids."*

Of the anhydrides of organic dibasic acids but very little is known; only the internal anhydrides, which cannot be formed except in those cases where the position of the two carboxyl groups in the molecule is stated to be favourable, have been investigated. But in some cases at least we may expect others formed in the same manner as those of the monobasic acids, namely by the co-operation of two molecules instead of the exercise of the two functions of the same molecule.

We may equally expect that when the dibasic acid has passed into a monobasic one, for instance by changing one of the acid functions into an ester or a salt, this will anyway yield an anhydride in the same manner as other monobasic acids.

Of some mixed anhydrides which are also esters we know, for instance, the ethyloxalylchloride but not the simple anhydrides. One of the chief methods of preparing the simple anhydrides is the one applied by GERHARDT in 1853, namely, the action of acid chlorides (mixed anhydrides) on salts. It is this method which, at any rate with oxalic acid, has at once yielded the desired product.

Dr. MOL allowed ethyloxalylchloride to act with the usual precautions on the potassium salt of acid ethyloxalate covered with ether and obtained a colourless liquid which distilled at 85°—90° under a pressure of less than 1 millimetre, solidified on cooling and then melted at 4°. The results of the elementary analysis and of the determination of the molecular weight agree with what is required by the desired anhydride



ethyloxalanhydride.

as does the decomposition by water. On being heated at the ordinary pressure it is decomposed with evolution of gas.

Dr. MOL obtained this substance in a still simpler manner by acting with oxychloride of phosphorus on an excess of potassium ethyl oxalate. The investigation is being continued with other dibasic acids.

Chemistry. — "*Thalictrum aquilegifolium, a hydrogen cyanide-yielding plant.*" By Dr. L. VAN ITALLIE. (Communicated by Prof. P. VAN ROMBURGH).

The communications from GUIGNARD (Compt. rend. de l'Acad. des Sciences du 24 Juillet 1905) as to the presence of a hydrogen cyanide-yielding glucoside in the leaves of Sambucus nigra L. and other varieties of elder have induced me to continue the experiments previously made in the same direction. I have been able to confirm the observations of GUIGNARD in every particular notwithstanding the figures which I found for the HCN-content are lower than those stated by him. This may, probably, be explained by the fact that I did not test the elder leaves until the beginning of September whilst GUIGNARD made his experiments in June.

From 100 grams of fresh leaves of Sambucus nigra I obtained 8,3 milligrs. and from 100 grams of Sambucus nigra var. laciniata 7,7 milligrs. of HCN. No HCN was obtained from 100 grams of Sambucus Ebulus.

The ornamental plant Thalictrum aquilegifolium (which appears to

grow wild in the environs of Nijmegen) appears, however, to be comparatively rich in HCN-yielding material.

If the leaves of this Ranunculaceus are crushed and digested with water for 12 hours at 30°—36° a hydrogen cyanide-containing distillate will be obtained on distillation.

The distillate from 100 grams of fresh leaves collected on Sept. 11 in the botanical garden of the Veterinary School yielded 248,8 milligrs. of AgCN = 50,2 milligrs. of HCN = 0,05 per cent. A volumetric experiment which showed 53 milligrs. of HCN confirmed this result.

A third experiment made with leaves, kindly forwarded to me from the Botanical Gardens at Groningen, gave 0,06 per cent of HCN in the distillate obtained from the same quantity of leaves.

I failed to obtain any HCN from the root of the plant and 142 grams of the fresh stem only yielded 4,4 milligrs. of HCN.

The leaves of Thalictrum aquilegifolium are therefore, comparatively rich in HCN-yielding material.

No HCN-containing distillate could be obtained from Thalictrum flavum, Thalictrum minus and Thalictrum glaucum.

Hydrogen cyanide could not be detected in the leaves in the free state. When fresh leaves were immersed in hot alcohol no HCN could be detected in the alcoholic distillate.

The hydrogen cyanide is formed during the digestion and is, therefore, most probably liberated from a glucoside by the action of an enzyme.

This enzyme is probably closely related to emulsin. I have obtained it, in an impure condition, by extracting the fresh, crushed leaves with water, and adding to the filtrate a large amount of alcohol. The precipitate so obtained was carefully dried; it very readily resolved amygdalin.

The glucoside present in Thalictrum aquilegifolium is not identical with amygdalin but is probably so with phaseolunatin isolated from Phaseolus lunatus by Dunstan and Henry (Proc. Royal Soc. LXXII, 482, 1903), because in the hydrogen cyanide-containing distillate acetone can be detected, but no benzaldehyde. The presence of the former was shown from the iodoform-reaction with ammonia and tincture of iodine and the solubility of freshly precipitated mercuric oxide in the distillate.

Owing to the small quantity of leaves at my disposal it was useless to attempt the isolation of the glucoside in the pure state. I intend doing so next year, and also to watch the development of the glucoside in the plant.

I may, however, state provisionally that this glucoside is either insoluble or at most very slightly soluble in cold alcohol. When the leaves, after being dried in an airbath at 80°, and then powdered, were extracted with cold alcohol, no HCN and acetone could be obtained by enzyme-action from the alcoholic residue.

When the extracted powder after being dried was mixed with water, and then brought in contact with the enzyme, the aqueous distillate showed abundant evidence of the presence of HCN and acetone.

Utrecht, September 25, 1905.

Chemistry. — Prof. P. VAN ROMBURGH presents a communication: *"On the action of ammonia and amines on formic esters of glycols and glycerol"* (II).

As the action of ammonia and amines on allyl formate (Proc. June 24 '05) had yielded such good results to me, I have also included in my research other formic esters, and I now communicate, briefly, the results obtained with the formates of some polyhydric alcohols.

If gaseous ammonia is allowed to act on the diformate of glycol it is first absorbed slowly with evolution of heat. If, when the action is over, the liquid is distilled, nothing passes over at the boiling point of the diformate (174°), but the temperature rises at once to the boiling point of glycol, and then gradually to that of formamide. A complete separation of the two substances, whose boiling point only differs about 20°, does not succeed with small quantities, and although it has been proved that the reaction takes place readily and almost quantitatively, formamide cannot be obtained pure in this way.

One gram of the diformate when mixed with 2 grams of dipropylamine gave a slow rise from 18° to 42°. The liquid being distilled the formate again seemed to have disappeared, and a fraction could be obtained at the boiling point of the glycol, and another at that of dipropylformamide.

With 1.8 gram of benzylamine, 1 gram of glycol diformate gave a slow rise from 18° to 80°. On distillation, the formate seemed to have disappeared and the glycol being distilled off, nearly the theoretical amount of benzylformamide was left in a pure condition.

If gaseous ammonia is allowed to act on the diformate of propanediol (1. 2), which I prepared by heating this glycol with formic acid, phenomena are noticed analogous to those in the case of glycol

diformate. After the action is over the ester has again disappeared, and a mixture of propanediol (1. 2) and formamide has formed.

7 grams of this diformate being mixed with 10 grams of piperidine the temperature rose from 20° to 120° and on fractionation it again appeared that the ester had been completely converted into propanediol, whilst the formylpiperidine, after a few distillations, could be separated in a fairly pure condition. The boiling point was a little too low, probably owing to traces of the glycol.

With 7 grams of benzylamine, the diformate of propanediol (1. 2.) gave a rise from 20° to 110°. On distillation the formed glycol passed over at about 190°. The residue which had been heated to about 250° (thermometer immersed in the liquid) solidified on cooling, and consisted of nearly pure benzylformamide. It may be distilled at about 295° with only slight decomposition. The distilled product had a faint odour of carbylamine, and melted at 59°. By recrystallisation the melting point rose to 61°.

If gaseous ammonia is passed into a mixture of formines of glycerol, such as is obtained for instance by boiling glycerol with formic acid, or heating with oxalic acid, and then removing the free formic acid by distillation in vacuo, it is absorbed with great evolution of heat. After expelling the excess of ammonia and distilling in vacuo a rich yield of almost pure formamide is obtained.

In one of my experiments 66 grams of formine (yielding 65 % of formic acid on saponification) was saturated with ammonia. In the first distillation 22 grams of formamide m.p. 0° and 17 grams dito m.p. — 2° were separated whilst 40 grams of glycerol remained in the distilling flask. The yield was therefore practically the theoretical one so that this method may be recommended for the rapid preparation of formamide in large quantities.

With pure triformine [1]) the action of ammonia is slower than with the above mentioned mixture. Triformine of glycerol eagerly absorbs gaseous dimethylamine with strong evolution of heat, and on distillation in vacuo a good yield of the dimethylformanide b. p. 153° is obtained. Piperidine gives with triformine a considerable rise in temperature (from 20° to 70°).

Dipropylamine forms with triformine, at first, two layers. After a little shaking (the temperature rose from 18° to 77°) the liquid becomes homogeneous, and by distillation in vacuo a good yield of the dipropylformamide formed could be readily obtained.

With diisobutylamine, triformine also gives two layers which do

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[1]) I hope to communicate about this substance, shortly.

not disappear on shaking for a while, but if the liquid was allowed to stand over night it became homogeneous, and on distillation in vacuo yielded diisobutylformamide.

Formic esters of unsaturated glycols also seem to react readily with amines, at least Mr. W. VAN DORSSEN, who is engaged in the Utrecht laboratory upon the study of the 3.4-dihydroxy-1.5-hexadiene

$$\begin{array}{l} CH_2 = CH - CH.OH \\ \qquad\qquad\quad | \\ CH_2 = CH - CH.OH \end{array}$$

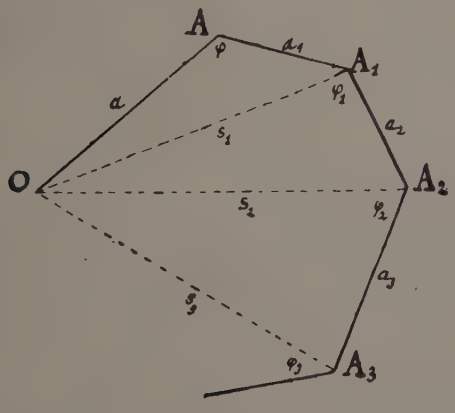
obtained, on mixing 1 gram of the diformate of this glycol with 1.3 gram of benzylamine, a rise in temperature from 18° to 65°, and after distilling off the glycol could readily isolate benzylformamide m. p. 61°.

Mathematics. — "*A local probability problem*". By Prof. J. C. KLUYVER.

The following problem was lately (Nature, July 27) proposed by Prof. PEARSON:

"A man starts from a point O, and walks l yards in a straight line; he then turns through any angle whatever, and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + dr$ from his starting point O."[1])

I find that the general solution of this problem depends upon the theory of BESSEL's functions, especially that in some particular cases it leads to the evaluation of certain definite integrals, involving these functions.



Let $OAA_1A_2A_3 \ldots A_{n-1}$ be the broken line, the n stretches of which need not be all of the same length. Then the shape of the figure, not its orientation in the plane, is wholly determined by the lengths $a, a_1, a_2, \ldots, a_{n-1}$ of the stretches, and by the magnitudes of the angles $\varphi, \varphi_1, \ldots, \varphi_{n-2}$, formed at the origin of each stretch a_k by the stretch itself and by the radius vector s_{k-1}.

[1]) Recently (Nature, August 10) Prof. PEARSON stated, that the solution for n very large was already virtually contained in a memoir on sound by Lord RAYLEIGH.

In a turning point the rambler takes his new direction at random; hence for any angle φ_k, all values between 0 and 2π have an equal chance, and the probability that those angles are respectively included within the intervals, φ_k, $\varphi_k + d\varphi_k$, is equal to the product

$$\frac{1}{(2\pi)^{n-1}} d\varphi\, d\varphi_1 \ldots d\varphi_{n-2}.$$

If we integrate this product over a region, determined by the condition that the nth radius vector s_{n-1} remains less than a given distance c, the result will be the required probability $W_n(c\,;aa_1a_2\ldots a_{n-1})$, that the ending point of the path lies within the distance c from the starting point O.[1])

The integration becomes less complicated, if we introduce in the usual way a discontinuous factor. Choosing a function $T(\varphi,\varphi_1,..,\varphi_{n-2})$ such, that it vanishes when $s_{n-1} > c$, and that it is equal to unity for $s_{n-1} < c$, to each of the variables φ_k we may give the whole range from 0 to 2π, and we have

$$W_n(c\,;aa_1\,..\,a_{n-1}) = \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} d\varphi d\varphi_1\,..\,d\varphi_{n-2}\, T(\varphi,\varphi_1,\ldots,\varphi_{n-2}).$$

For the function T we may take WEBER's discontinuous integral, that is, we may put

$$T(\varphi,\varphi_1,..,\varphi_{n-2}) = c\int_0^{\infty} J_1(uc)J_0(us_{n-1})du,$$

the integral being equal to zero or to unity according to s_{n-1} being larger or smaller than c.

This choice of the factor T makes a good deal of reduction possible.

If we consider the side c of a triangle as a function of the sides a and b and of the inclosed angle C, the relation holds

$$J_0(ua)\,J_0(ub) = \frac{1}{2\pi}\int_0^{2\pi} J_0(uc)\,dC,$$

and this formula can be repeatedly used in reducing the integral $W_n(c\,;aa_1\ldots a_{n-1})$.

So we get successively

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[1]) In the case $n=2$, we have, supposing $a + a_1 > c > a - a_1$, $W_2(c;aa_1) = \frac{1}{\pi}\arccos\frac{a^2+a_1^2-c^2}{2aa_1}$. Of course for $c > a+a_1$ W_2 becomes equal to unity and it is zero for $a - a_1 > c$.

$$J_0(us_{n-2})\,J_0(ua_{n-1}) = \frac{1}{2\pi}\int_0^{2\pi} J_0(us_{n-1})\,d\varphi_{n-2}\,,$$

$$J_0(us_{n-3})\,J_0(ua_{n-2}) = \frac{1}{2\pi}\int_0^{2\pi} J_0(us_{n-2})\,d\varphi_{n-1}\,,$$

. . . . . .

$$J_0(ua)\,J_0(ua_1) = \frac{1}{2\pi}\int_0^{2\pi} J_0(us_1)\,d\varphi\,,$$

and consequently

$$W_n(c\,;\,aa_1\ldots a_{n-1}) = c\int_0^{\infty} J_1(uc)\,J_0(ua)\,J_0(ua_1)\ldots J_0(ua_{n-1})\,du.$$

From this result we infer, that the probability sought for is of a rather intricate character. The $n+1$ functions J are oscillating functions, and have their signs altering in an irregular manner as the variable u increases. Hence even an approximation of the integral is not easily found, and as a solution of PEARSON's problem it is little apt to meet the requirements of the proposer.

From a mathematical point of view the integral presents some interest. In fact, if we consider it as a function of c, it is readily seen to be continuous and finite for all real values of c, and the same holds for a certain number of derivatives with respect to c, but a closer inspection shows, that this analytic expression, regularly built up as it is, represents in different intervals different analytic functions. To make good this assertion, we have only to remember that the integral stands for the probability required in PEARSON's problem. Hence we know beforehand, that it always must be positive and increasing with c, but that it never surpasses 1, this upper limit being actually reached as soon as c becomes greater than $a+a_1+\ldots+a_{n-1}$. Moreover, if we suppose $a > a_1+a_2+\ldots+a_{n-1}$, the inequality $a > c+a_1+a_2\ldots+a_{n-1}$ is possible for small values of c. And if the latter inequality holds, the rambler of PEARSON's problem necessarily arrives outside the circle with radius c, and the probability is zero.

Thus, by solving the problem, we have found

$$\left.\begin{array}{l} c > a+a_1\ldots+a_{n-1}\ ,\ldots 1 \\ a > c+a_1\ldots+a_{n-1}\ ,\ldots 0 \end{array}\right\} = c\int_0^{\infty} J_1(uc)\,J_0(ua)\,J_0(ua_1)\ldots J_0(ua_{n-1})\,du\,,$$

quite independently of the number of the J_0-functions, showing

thereby at the same time, that the continuous analytic expression cannot be regarded as a single analytic function.

The same still holds for values of c, not fulfilling one of the above inequalities, though the integral is then continuously varying with c.

So for instance in the case $n = 3$, taking the stretches $a > a_1 > a_2$ in such a manner, that a triangle is possible having these sides, I am led to conclude from the discontinuities of the first derivative that in each of the following intervals

I	$a_1 + a_2 - a > c > 0$	*IV*	$a + a_1 + a_2 > c > a + a_1 - a_2$
II	$a - a_1 + a_2 > c > a_1 + a_2 - a$	*V*	$c > a + a_1 + a_2$
III	$a + a_1 - a_2 > c > a - a_1 + a_2$		

a distinct analytic function is defined by the integral.

Some further remarks may be made. On integrating by parts we find

$$W_n(c; aa_1 \ldots a_{n-1}) = 1 - a\int_0^\infty J_1(ua)\, J_0(uc)\, J_0(ua_1) \ldots J_0(ua_{n-1})\, du$$

$$- a_1 \int_0^\infty J_1(ua_1)\, J_0(ua)\, J_0(uc) \ldots J_0(ua_{n-1})\, du$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

or what is the same:

$$1 = W_n(c; aa_1 \ldots a_{n-1}) + W_n(a; ca_1 \ldots a_{n-1}) + W_n(a_1; ac \ldots a_{n-1}) + \ldots\ldots$$

Dividing both sides of the equation by $n+1$ we may interpret the coming relation as follows: $n+1$ equal or unequal stretches being given, if n of them, taken at random, are put together to a broken line, according to the rules of PEARSON's problem, the probability is equal to $\frac{1}{n+1}$, that the distance between the extremities of this broken line is less than the stretch that was left out.

And from the same equation we deduce in the very particular case: $c = a = a_1 \ldots = a_{n-1}$

$$W_n(a; a^n) = \frac{1}{n+1},$$

or: the rambler of PEARSON's problem after walking along n equal stretches has the chance $\frac{1}{n+1}$ to find himself within a stretches' length from his starting point.

In the most general case of the problem I cannot give a practical solution; something however can be done, in the case: n very large, all stretches equal, treated already by Lord RAYLEIGH.

Putting $na = L$, $c = \frac{L}{a}$, we have

$$W_n(c; a^n) = W_n(c/L) = \int_0^\infty J_1(u)\, J_0\left(\frac{au}{n}\right)^n du.$$

Now by raising to the nth power the ordinary power series for $J_0\left(\frac{au}{n}\right)$ we get

$$J_0\left(\frac{au}{n}\right)^n = 1 + \sum_{k=1}^{k=\infty} \frac{(-1)^k}{k!} \cdot \left(\frac{au}{2}\right)^{2k} \cdot \frac{S_k(n)}{k!\, n^{2k}},$$

where $S_k(n)$ stands for the sum of squares of the coefficients of the expansion $(u_1 + u_2 + \ldots u_n)^k$, so that

$$\frac{S_1(n)}{1!\, n^2} = \frac{1}{n}, \quad \frac{S_2(n)}{2!\, n^4} = \frac{1}{n^2} - \frac{1}{2n^3}, \quad \frac{S_3(n)}{3!\, n^6} = \frac{1}{n^3} - \frac{3}{2n^4} + \frac{2}{3n^5}.$$

Generally supposing n very large we may put approximately

$$\frac{S_k(n)}{k!\, n^{2k}} = \frac{1}{n^k},$$

and, substituting, we find that this approximation leads to the supposition

$$J_0\left(\frac{au}{n}\right)^n = e^{-\frac{a^2u^2}{4n}}.$$

For small values of u the approximation is good enough. It is true both functions behave quite differently when u becomes very large, but as they are rather rapidly converging to zero, the actual amount of their difference can be neglected. In particular I find that the integral

$$\int_n^\infty J_1(u)\, J_0\left(\frac{au}{n}\right)^n du$$

is of an order of smallness certainly higher than that of the expression

$$\frac{2n}{n-2} \cdot \left(\frac{2}{\pi}\right)^{\frac{n+1}{2}} \cdot \left(\frac{1}{a}\right)^{\frac{n}{2}},$$

while the order of smallness of the integral

$$\int_n^\infty J_1(u) e^{-\frac{a^2u^2}{4n}} du$$

is that of the expression

$$e^{-\frac{1}{4}a^2n}\,(J_0(n) - J_0(q)), \quad q > n.$$

Hence if only a be rather greater than unity, both integrals cannot have an appreciable difference and we may put

$$W_n(c/L) = \int_0^\infty J_1(u) e^{-\frac{u^2 a^2}{4n}} du = 1 - e^{-\frac{n}{a^2}} = 1 - e^{-n\frac{c^2}{L^2}}.$$

From this result it is evident, that $W_n(c/L)$ for n very large is always nearly unity. The rambler, walking along a very great number of very short stretches will almost certainly arrive in the neighbourhood of his starting point.

Putting $c = \frac{1}{n} L$, we find $W_n(c/L) = 1 - e^{-\frac{1}{n}} = \frac{1}{n} - \frac{1}{2n^2} \ldots$, a result nearly equal to the true value $\frac{1}{n+1}$.

Returning to the general expression for $W_n(c; aa_1 \ldots a_{n-1})$ we observe the possibility of differentiating the integral with respect to c in the usual way a number of $2m$ times, provided $2m < \frac{n+1}{2}$.

Suppossing $c > a + a_1 + \ldots + a_{n-1}$ and putting

$$J_0(ua) J_0(ua_1) \ldots J_0(ua_{n-1}) = f(u),$$

we deduce by differentiation

$$1 = c \int_0^\infty J_1(cu) f(u) du,$$

$$0 = \int_0^\infty u J_0(cu) f(u) du \quad , \quad 0 = \int_0^\infty u^2 J_1(cu) f(u) du,$$

$$0 = \int_0^\infty u^3 J_0(cu) f(u) du \quad , \quad 0 = \int_0^\infty u^4 J_1(cu) f(u) du,$$

. . . .

$$0 = \int_0^\infty u^{2m-1} J_0(cu) f(u) du \quad , \quad 0 = \int_0^\infty u^{2m} J_1(cu) f(u) du.$$

These equations allow us to introduce into the integral a new BESSEL function, the function $J_{2m+1}(u)$. For $J_{2m+1}(u)$ is connected with $J_0(u)$ and $J_1(u)$ by the relation

$$J_{2m+1}(u) = P_{0,2m}(u) J_1(u) - P_{1,2m-1}(u) J_0(u),$$

where

$$P_{0,2m}(u) = \frac{1}{u^{2m}}(b_0 + b_2 u^2 + \ldots + b_{2m} u^{2m})$$

and

$$P_{1,2m-1}(u) = \frac{1}{u^{2m}}(b_1 u + b_3 u^3 + \ldots + b_{2m-1} u^{2m-1})$$

are a pair of SCHLÄFLI's polynomials.

Using this relation we obtain

$$b_0 = c^{2m+1} \int_0^\infty u^{2m} J_{2m+1}(uc) f(u)\, du,$$

and as

$$b_0 = \underset{u=0}{Lim}\, u^{2m} P_{0,2m}(u) = 2^{2m}\, m!$$

we have

$$2^{2m}\, m! = c^{2m+1} \int_0^\infty u^{2m} J_{2m+1}(uc) J_0(ua) J_0(ua_1) \ldots J_0(ua_{n-1})\, du$$

with the conditions

$$c > a + a_1 + \ldots + a_{n-1} \quad , \quad m < \frac{n+1}{4}.$$

Evidently the value of the integral would be zero, if instead of the first of these conditions the condition

$$a > c + a_1 + \ldots + a_{n-1}$$

was satisfied.

In the same manner we might differentiate and also integrate with respect to one or to several of the parameters a. This leads for instance to the following results

$$n \text{ even}: \qquad 0 = \int_0^\infty J_1(uc) J_1(ua) J_1(ua_1) \ldots J_1(ua_{n-1})\, du$$

$$n \text{ odd}: \qquad 0 = \int_0^\infty u\, J_1(uc) J_1(ua) J_1(ua_1) \ldots J_1(ua_{n-1})\, du\,.$$

$$c > a + a_1 + a_2 + \ldots + a_{n-1}.$$

Still other results present themselves when PEARSON's problem is slightly modified. Again putting

$$J_0(ua) J_0(ua_1) \ldots J_0(ua_{n-1}) = f(u)$$

and writing ϱ for c, we get by differentiation with respect to ϱ

$$W_n(d\Omega) = \frac{1}{2\pi}\,\varrho\,d\varrho\,d\theta\int_0^\infty u\,J_0(u\varrho)\,f(u)\,du,$$

and here $W_n(d\Omega)$ means the probability that the ending point of the broken line falls on a given element $d\Omega$ of the plane, the polar coordinates of which are ϱ, θ.

By integrating over a given finite region we may deduce the probability that the rambler reaches that region [1]).

First let the region be a rectangle R, and let the rectangular coordinates of its vertices be $\pm p$, $\pm q$, then we find for the corresponding probability

$$W_n(R) = \frac{1}{2\pi}\int_0^\infty u\,f(u)\,du\int_{-p}^{+p}d\xi\int_{-q}^{+q}d\eta\,J_0(u\sqrt{\xi^2+\eta^2}).$$

Now we have

$$J_0(u\sqrt{\xi^2+\eta^2}) = \frac{1}{2\pi}\int_0^{2\pi}\cos(u\,\xi\,\cos\alpha)\cos(u\,\eta\,\sin\alpha)\,d\alpha,$$

and therefore, effectuating the integrations with respect to ξ and to η,

$$W_n(R) = \frac{4}{\pi^2}\int_0^\infty u\,f(u)\,du\int_0^{\frac{\pi}{2}}\frac{\sin(pu\cos\alpha)\sin(qu\sin\alpha)}{u^2\sin\alpha\cos\alpha}\,d\alpha.$$

A somewhat simpler expression is found, if changing the variables we pass from u and α to

$$v = u\cos\alpha,$$
$$w = u\sin\alpha.$$

Then the probability $W_n(R)$ is expressed as follows:

$$W_n(R) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty dv\,dw\,\frac{\sin pv}{v}\cdot\frac{\sin qw}{w}\cdot f(\sqrt{v^2+w^2}).$$

Again an evaluation of this double integral is generally not practicable, but the problem itself gives the value of the integral, if both

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[1]) If this region is a circle with radius c, the centre of which lies at a distance b from the starting point O, we have at once

$$W_{n+1}(c;\,baa_1\ldots a_{n-1}) = c\int_0^\infty J_1(uc)\,J_0(ub)\,J_0(ua)\,J_0\,ua_1)\ldots J_0(ua_{n-1})\,du$$

for the probability, that the path ends inside the circle.

the coordinates p, q are surpassing the total length of the path. Then the probability becomes a certainty and it follows that

$$\frac{\pi^2}{4} = \int_0^\infty \int_0^\infty dv\, dw \, \frac{\sin pv}{v} \cdot \frac{\sin qw}{w} \cdot f(\sqrt{v^2 + w^2})$$

with the condition

$$p \text{ and } q > a + a_1 + \ldots + a_{n-1}.$$

In the general case of the rectangle the probability $W_n(R)$ is independent of q, as soon as its length is superior to that of the path.

Assuming this to be the case, we remark that the value of the slightly transformed integral

$$W_n(R) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty dv\, dw \, \frac{\sin pv}{v} \cdot \frac{\sin w}{w} \cdot f\left(\sqrt{v^2 + \frac{w^2}{q^2}}\right)$$

remains unaltered, when q increases indefinitely, and we conclude that

$$\underset{q=\infty}{Lim}\, W_n(R) = \frac{4}{\pi^2} \int_0^\infty \frac{\sin pv}{v} f(v)dv \cdot \int_0^\infty \frac{\sin w}{w}\, dw = \frac{2}{\pi} \int_0^\infty \frac{\sin pv}{v} f(v)dv.$$

Thus we have solved another modification of PEARSON's problem, for half the result, added to $\frac{1}{2}$, expresses the probability

$$W_n(F) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin pv}{v} f(v)\, dv\,,$$

that the rambler, starting on his walk at a distance p of a straight frontier F, after walking along n stretches, will arrive at that side of the frontier he came from [1]).

As before we are enabled in a particular case by the problem itself to assign the value of the integral. If we suppose that the rambler cannot reach the frontier, that is, if we take

$$p > a + a_1 + \ldots + a_{n-1}\,,$$

the probability becomes a certainty and we find

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1) Obviously the probability $W_n(F)$ might have been derived from the probability $W_{n+1}(\omega + p\,;\, \omega\, a\, a_1 \ldots a_{n-1})$ by making ω indefinitely large. Therefore we may conclude that

$$\underset{\omega=\infty}{Lim}\,(\omega+p) \int_0^\infty J_1(u\overline{\omega+p})\, J_0(u\omega) f(u)du = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin vp}{v} f(v)dv.$$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin vp}{v} J_0(va) J_0(va_1) \ldots J_0(va_{n-1})\, dv .$$

In the case $n = 1$, this is a known result to which another may be added, if we take $a > p$. When the single stretch a is inclined to the frontier under an angle less than

$$arc\ sin \frac{p}{a},$$

the rambler remains at the same side and, all directions of the stretch being equally possible, we have

$$W_1(F) = \frac{1}{\pi}\left(\frac{\pi}{2} + arc\ sin \frac{p}{a}\right),$$

hence

$$arc\ sin \frac{p}{a} = \int_0^\infty \frac{\sin vp}{v} J_0(va)\, dv .$$

Mathematics. — *"A definite integral of* KUMMER". By Prof. W. KAPTEYN.

In CRELLE's Journal, Vol. 17, KUMMER has determined the value of the integral

$$U_p = \int_0^\infty e^{-x-\frac{b^2}{x}} x^p\, dx ,$$

supposing b^2 to represent a positive quantity and p not an integer. He finds:

$$U_p = \Gamma(p+1) f(-p, b^2) + \Gamma(-p-1)\, b^{2p+2} f(p+2, b^2),$$

where

$$f(p,x) = 1 + \frac{x}{1!p} + \frac{x^2}{2!p(p+1)} + \frac{x^3}{3!p(p+1)(p+2)} + \ldots$$

$$+ \ldots = \sum_{s=0}^{\infty} \frac{x^s}{s!p(p+1)..(p+s-1)} .$$

In the following pages we propose to study this integral for the case that p represents a positive integer, and at the same time to show that there is a simple connection between this integral and the integral

$$V_p = \int_b^\infty e^{-x-\frac{b^2}{x}} x^p\, dx,$$

where b is supposed to be positive.

It is rational to put in the integral of KUMMER

$$p = n - \varepsilon,$$

assuming n to be an integer and ε an arbitrary infinitesimal, and then to determine the limit for $\varepsilon = 0$.

Let us therefore examine the limit of

$$U_{n-\varepsilon} = \Gamma(n+1-\varepsilon) f(-n+\varepsilon, b^2) + \Gamma(-n-1+\varepsilon) b^{2n+2-2\varepsilon} f(n+2-\varepsilon, b^2)$$

for $\varepsilon = 0$.

Suppose

$$\Gamma(n+1-\varepsilon) = A_0 + A_1 \varepsilon + A_2 \varepsilon^2 \ldots$$

$$f(-n+\varepsilon, b^2) = \frac{B_0}{\varepsilon} + B_1 + B_2 \varepsilon + \ldots$$

$$\Gamma(-n-1+\varepsilon) = \frac{C_0}{\varepsilon} + C_1 + C_2 \varepsilon + \ldots$$

$$b^{2n+2-2\varepsilon} = D_0 + D_1 \varepsilon + D_2 \varepsilon^2 + \ldots$$

$$f(n+2-\varepsilon, b^2) = E_0 + E_1 \varepsilon + E_2 \varepsilon^2 + ,$$

then

$$U_{n-\varepsilon} = \frac{A_0 B_0 + C_0 D_0 E_0}{\varepsilon} + [A_1 B_0 + A_0 B_1 + C_1 D_0 E_0 + C_0 D_1 E_0 + C_0 D_0 E_1) + .$$

and the limit

$$U_n = A_1 B_0 + A_0 B_1 + C_1 D_0 E_0 + C_0 D_1 E_0 + C_0 D_0 E_1,$$

for we shall see that

$$A_0 B_0 + C_0 D_0 E_0 = 0.$$

Let us now determine the various coefficients.

First we have

$$\Gamma(n+1-\varepsilon) = \Gamma(n+1) - \varepsilon \Gamma(n+1) \frac{\Gamma'(n+1)}{\Gamma(n+1)} + \ldots$$

or if we put

$$\frac{\Gamma'(x)}{\Gamma(x)} = \psi(x),$$

$$\Gamma(n+1-\varepsilon) = n! [1 - \varepsilon \psi(n+1) + \ldots],$$

thus

$$A_0 = n!,$$

$$A_1 = - n! \psi(n+1).$$

To find B_0 and B_1 we write

$$f(-n+\varepsilon, b^2) = \sum_{s=0}^{\infty} \frac{b^{2s}}{s!(-n+\varepsilon)(-n+1+\varepsilon)\ldots(-n+s-1+\varepsilon)} +$$

$$+ \frac{1}{\varepsilon} \sum_{s=0}^{\infty} \frac{b^{2n+2+2s}}{(n+s+1)!(-n+\varepsilon)(-n+1+\varepsilon)\ldots(-1+\varepsilon)(1+\varepsilon)\ldots(s+\varepsilon)}$$

If

$$\frac{1}{(-n+\varepsilon)(-n+1+\varepsilon)\ldots(-1+\varepsilon)(1+\varepsilon)\ldots(s+\varepsilon)} = \lambda(\varepsilon) = \lambda(0)\left[1+\varepsilon\frac{\lambda'(0)}{\lambda(0)}+\ldots\right],$$

then we easily find

$$\lambda(0) = \frac{(-1)^n}{n!s!},$$

and

$$\frac{\lambda'(0)}{\lambda(0)} = \frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{1}-\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{s}\right) = \psi(1+n)-\psi(1+s),$$

therefore

$$B_0 = \frac{(-1)^n b^{2n+2}}{n!}\sum_{s=0}^{\infty}\frac{b^{2s}}{s!\,(n+s+1)!} = \frac{(-1)^n b^{2n+2}}{n!\,(n+1)!}f(n+2,b^2)$$

$$B_1 = \frac{1}{n!}\sum_{s=0}^{n}\frac{(-1)^s(n-s)!}{s!}b^{2s} + \frac{(-1)^n b^{2n+2}}{n!\,(n+1)!}\psi(1+n)f(n+2,b^2) -$$

$$-(-1)^n\frac{b^{2n+2}}{n!}\sum_{s=0}^{\infty}\frac{\psi(1+s)b^{2s}}{s!(n+s+1)!}.$$

For the evaluation of C_0 and C_1 we have

$$\Gamma(-n-1+\varepsilon) = \frac{1}{-n-1+\varepsilon}\Gamma(-n+\varepsilon),$$

$$\Gamma(-n+\varepsilon) = \frac{1}{-n+\varepsilon}\Gamma(-n+1+\varepsilon),$$

. . . . . .

$$\Gamma(-2+\varepsilon) = \frac{1}{-2+\varepsilon}\Gamma(-1+\varepsilon),$$

$$\Gamma(-1+\varepsilon) = \frac{1}{-1+\varepsilon}\Gamma(\varepsilon),$$

so

$$\Gamma(-n-1+\varepsilon) = \frac{(-1)^{n+1}}{(n+1-\varepsilon)(n-\varepsilon)\ldots(2-\varepsilon)(1-\varepsilon)}\Gamma(\varepsilon).$$

Assuming

$$\frac{1}{(n+1-\varepsilon)(n-\varepsilon)\ldots(2-\varepsilon)(1-\varepsilon)} = \mu(\varepsilon) = \mu(0)\left[1+\varepsilon\frac{\mu'(0)}{\mu(0)}+\ldots\right],$$

then

$$\mu(0) = \frac{1}{(n+1)!},$$

$$\frac{\mu'(0)}{\mu(0)} = \frac{1}{n+1}+\frac{1}{n}+\ldots+\frac{1}{2}+1 = \psi(n+2)-\psi(1),$$

whilst

$$\Gamma(\varepsilon) = P(\varepsilon)+Q(\varepsilon)$$

$$= \frac{1}{\varepsilon} - \frac{1}{1+\varepsilon} + \frac{1}{2\,!}\frac{1}{2+\varepsilon} - \frac{1}{3\,!}\frac{1}{3+\varepsilon} + \ldots + Q(\varepsilon)$$

$$= \frac{1}{\varepsilon} + Q(0) - 1 + \frac{1}{2\,!2} - \frac{1}{4\,!3} + \frac{1}{4\,!4} - \ldots$$

If we no notice that

$$Q(0) = \int_1^\infty \frac{e^{-x}}{x}\,dx = -\int_0^{e^{-1}} \frac{dy}{lgy} = -\,li(e^{-1})$$

and that out of the well known formula

$$li(e^{-v}) = -\,\psi(1) + lgx - x + \frac{x^2}{2\,!2} - \frac{x^3}{3\,!3} + \frac{x^4}{4\,!4} - \ldots$$

follows

$$li(e^{-1}) = -\,\psi(1) - 1 + \frac{1}{2\,!2} - \frac{1}{3\,!3} + \frac{1}{4\,!4} - \ldots,$$

then it is evident that

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} + \psi(1) + \ldots,$$

from which ensues

$$C_0 = \frac{(-1)^{n+1}}{(n+1)\,!},$$

$$C_1 = \frac{(-1)^{n+1}}{(n+1)\,!}\,\psi(n+2).$$

Moreover we find

$$b^{2n+2-2\varepsilon} = b^{2n+2}(e^{lgb})^{-2\varepsilon} = b^{2n+2}[1-2\varepsilon lgb + \ldots],$$

so

$$D_0 = b^{2n+2}$$

$$D_1 = -\,2b^{2n+2}lgb$$

and finally

$$f(n+2-\varepsilon, b^2) = \sum_{s=0}^{\infty} \frac{b^{2s}}{s\,!(n+2-\varepsilon)(n+3-\varepsilon)\ldots(n+s+1-\varepsilon)}.$$

If again we put

$$\frac{1}{(n+2-\varepsilon)(n+3-\varepsilon)\ldots(n+s+1-\varepsilon)} = r(\varepsilon) = r(0)\left[1+\varepsilon\frac{r'(0)}{r(0)} + \ldots\right],$$

we find

$$r(0) = \frac{1}{(n+2)(n+3)\ldots(n+s+1)}$$

$$\frac{r'(0)}{r(0)} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots\frac{1}{n+s+1} = \psi(n+s+2) - \psi(n+2),$$

so that

$$E_0 = f(n+2, b^2),$$

$$E_1 = -\psi(n+2) f(n+2, b^2) + \sum_{s=0}^{\infty} \frac{\psi(n+s+2) b^{2s}}{s!(n+2)\ldots(n+s+1)}.$$

With the aid of these values we find

$$A_0 B_0 + C_0 D_0 E_0 = 0,$$

$$A_1 B_0 + A_0 B_1 = \sum_{s=0}^{n} (-1)^s \frac{(n-s)!}{s!} b^{2s} - (-1)^n b^{2n+2} \sum_{s=0}^{\infty} \frac{\psi(1+s) b^{2s}}{s!\,(n+s+1)!},$$

$$C_1 D_0 E_0 + C_0 D_1 E_0 + C_0 D_0 E_1 =$$

$$= \frac{(-1)^n}{(n+1)!} b^{2n+2} \left[2 \lg b\, f(n+2, b^2) - \sum_{s=0}^{\infty} \frac{\psi(n+s+2) b^{2s}}{s!(n+2)(n+3)\ldots(n+s+1)} \right],$$

hence

$$U_n = \sum_{s=0}^{n} (-1)^s \frac{(n-s)!}{s!} b^{2s} +$$

$$+ (-1)^n b^{2n+2} \sum_{s=0}^{\infty} \frac{b^{2s}}{s!\,(s+n+1)!} [2 \lg b - \psi(1+s) - \psi(n+s+2)] \ldots (1)$$

Let us now determine U_n in another way to give this result another form. To this end we differentiate the equation

$$U_n = \int_0^{\infty} e^{-x-\frac{b^2}{x}} x^n \, dx,$$

we then get

$$-\frac{1}{2b} \frac{dU_n}{db} = \int_0^{\infty} e^{-x-\frac{b^2}{x}} x^{n-1} \, dx. \quad \ldots \ldots \quad (a)$$

$$-\frac{1}{2b} \frac{d^2 U_n}{db^2} + \frac{1}{2b^2} \frac{dU_n}{db} = -2b \int_0^{\infty} e^{-x-\frac{b^2}{x}} x^{n-2} \, dx.$$

Out of the identity

$$x^n \, d\left(e^{-x-\frac{b^2}{x}}\right) = -e^{-x-\frac{b^2}{x}} x^n \, dx + b^2 e^{-x-\frac{b^2}{x}} x^{n-2} \, dx,$$

we moreover deduce by integrating between the limits 0 and ∞

$$-n \int_0^{\infty} e^{-x-\frac{b^2}{x}} x^{n-1} \, dx = -\int_0^{\infty} e^{-x-\frac{b^2}{x}} x^n \, dx + b^2 \int_0^{\infty} e^{-x-\frac{b^2}{x}} x^{n-2} \, dx,$$

hence we find for U_n the differential equation

$$\frac{d^2 U_n}{db^2} - \frac{2n+1}{b} \frac{dU_n}{db} - 4 U_n = 0. \quad \ldots \ldots \quad (2)$$

This differential equation we also find if we put $x = 2i\,b$ and $v = \frac{U_n}{b^{n+1}}$ in BESSEL's equation:

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} + \left(1 - \frac{(n+1)^2}{x^2}\right)v = 0,$$

therefore

$$U_n = b^{n+1}\,[A\,I^{n+1}(2i\,b) + B\,Y^{n+1}(2i\,b)].$$

In order to determine the constants properly I notice that the integral U_n for $b = 0$ is equal to $n!$ and vanishes for $b = \infty$; moreover we find

for $b = 0$ $\qquad b^{n+1}\,I^{n+1}(2i\,b) = 0,$

„ „ $\qquad b^{n+1}\,Y^{n+1}(2i\,b) = -(-i)^{n+1}\frac{n!}{\pi},$

for $b = \infty$ $\qquad b^{n+1}\,I^{n+1}(2i\,b) = \frac{b^{n+1}}{2\sqrt{\pi b}}\,e^{2b + \frac{(n+1)i\pi}{2}},$

„ „ $\qquad b^{n+1}\,Y^{n+1}(2i\,b) = \frac{b^{n+1}}{2\sqrt{\pi b}}\,e^{2b + \frac{(n+2)i\pi}{2}},$

thus

$$\pi = -(-i)^{n+1}\,B,$$

$$0 = A + B\,e^{\frac{i\pi}{2}},$$

and finally

$$U_n = \pi i^{n+2}\,b^{n+1}\,[I^{n+1}(2i\,b) + i\,Y^{n+1}(2i\,b)] = \pi i^{n+2}\,b^{n+1}\,H_1{}^{n+1}(2i\,b)\,^1)\,..(3$$

That this value and the value (1) agree is easy to prove. For according to definition [1]) we find:

$$\pi\,Y^{n+1}(2i\,b) = 2\,I^{n+1}(2i\,b)\left(lg\,b + \frac{i\pi}{2}\right) - \left(\frac{1}{ib}\right)^{n+1}\sum_{s=0}^{n}\frac{(-1)^s(n-s)!}{s!}b^{2s} -$$

$$-(i\,b)^{n+1}\sum_{s=0}^{\infty}\frac{b^{2s}}{s!\,(s+n+1)!}\,[\psi(s+1) + \psi(s+n+2)],$$

from which ensues when we multiply by $i^{n+3}\,b^{n+1}$

$$\pi i^{n+2}\,b^{n+1}\,[I^{n+1}(2i\,b) + i\,Y^{n+1}(2i\,b)] = 2i^{n+3}\,b^{n+1}\,lg\,b\,I^{n+1}(2i\,b) +$$

$$+\sum_{s=0}^{n}\frac{(-1)^s(n-s)!}{s!}b^{2s} - (-1)^n\,b^{2n+2}\sum_{s=0}^{\infty}\frac{b^{2s}}{s!(s+n+1)!}[\psi(s+1)+\psi(s+n+2)].$$

By

$$I^{n+1}(2i\,b) = (i\,b)^{n+1}\sum_{s=0}^{\infty}\frac{b^{2s}}{s!\,(s+n+1)!},$$

[1]) NIELSEN, Handbuch der Cylinderf. page 16.

the second member of the former equation becomes the second member of the equation (1).

Let us now examine

$$V_n = \int_b^\infty e^{-x-\frac{b^2}{x}} x^n dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Here too we can find a differential equation satisfied by this integral. By differentiating we find

$$-\frac{1}{2b}\frac{dV_n}{db} = \int_b^\infty e^{-x-\frac{b^2}{x}} x^{n-1} dx + \frac{1}{2} b^{n-1} e^{-2b} \quad . \quad . \quad . \quad . \quad (b)$$

$$-\frac{1}{2b}\frac{d^2 V_n}{db^2} + \frac{1}{2b^2}\frac{dV^n}{db} =$$

$$= -2b \int_b^\infty e^{-x-\frac{b^2}{x}} x^{n-2} dx - 2b^{n-1} e^{-2b} + \frac{n-1}{2} b^{n-2} e^{-2b} \quad . \quad . \quad (c)$$

whilst the integration of the identity

$$x^n d\left(e^{-x-\frac{b^2}{x}}\right) = -e^{-x-\frac{b^2}{x}} x^n dx + b^2 e^{-x-\frac{b^2}{x}} x^{n-2} dx$$

between the limits b and ∞ furnishes

$$-b^n e^{-2b} - n \int_b^\infty e^{-x-\frac{b^2}{x}} x^{n-1} dx = -\int_b^\infty e^{-x-\frac{b^2}{x}} x^n dx + b^2 \int_b^\infty e^{-x-\frac{b^2}{x}} x^{n-2} dx .$$

So we find for V_n the differential equation

$$\frac{d^2 V_n}{db^2} - \frac{2n+1}{b}\frac{dV_n}{db} - 4V_n = (n+1) b^{n-1} e^{-2b} . \quad . \quad . \quad . \quad (5)$$

If we now write the equations (a) and (b)

$$\frac{dU_n}{db} = -2b\, U_{n-1} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\frac{dV_n}{db} = -2b\, V_{n-1} - b^n e^{-2b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

it is easy to find out of (6) and (2)

$$U_n = -\frac{b}{2}\frac{dU_{n-1}}{db} + n\, U_{n-1}$$

and likewise out of (7) and (5)

$$V_n = -\frac{b}{2}\frac{dV_{n-1}}{db} + n\, V_{n-1} + \frac{b^n}{2} e^{-2b} .$$

Out of the last two equations we deduce the recurrent relation

$$V_n - \tfrac{1}{2} U_n = -\frac{b}{2}\frac{d}{db}[V_{n-1} - \tfrac{1}{2} U_{n-1}] + n[V_{n-1} - \tfrac{1}{2} U_{n-1}] + \frac{b^n}{2} e^{-2b} \ldots (8)$$

by which we can reduce the evaluation of $V_n - \frac{1}{2} U_n$ to that of $V_0 - \frac{1}{2} U_0$.

Let us now determine the value of $V_0 - \frac{1}{2} U_0$. To this end we start from the equation (c); this becomes for $n = 0$

$$-\frac{1}{2b}\frac{d^2 V_0}{db^2} + \frac{1}{2b^2}\frac{dV_0}{db} = -2b \int_b^\infty e^{-x-\frac{b^2}{x}} \frac{dx}{x^2} - \frac{2}{b} e^{-2b} - \frac{1}{2b^2} e^{-2b}.$$

By substituting in this integral $\frac{b^2}{z}$ for x we find

$$\int_b^\infty e^{-x-\frac{b^2}{x}} \frac{dx}{x^2} = \frac{1}{b^2} \int_0^6 e^{-z-\frac{b^2}{z}} dz = \frac{1}{b^2}(U_0 - V_0),$$

hence the preceding equation becomes

$$U_0 = \frac{1}{4}\frac{d^2 V_0}{db^2} - \frac{1}{4b}\frac{dV_0}{db} + V_0 - e^{-2b}\left(1 + \frac{1}{4b}\right).$$

By subtracting from this according to (5)

$$0 = \frac{1}{4}\frac{d^2 V_0}{db^2} - \frac{1}{4b}\frac{dV_0}{db} - V_0 - \frac{e^{-2b}}{4b},$$

we find

$$V_0 - \frac{1}{2} U_0 = \frac{1}{2} e^{-2b}. \quad \ldots \ldots \quad (9)$$

With the aid of equation (8) we get:

$$V_1 - \frac{1}{2} U_1 = \left(b + \frac{1}{2}\right) e^{-2b},$$

$$V_2 - \frac{1}{2} U_2 = \left(\frac{3b^2}{2} + 2b + 1\right) e^{-2b},$$

$$V_3 - \frac{1}{2} U_3 = (2b^3 + 5b^2 + 6b + 3) e^{-2b},$$

$$V_4 - \frac{1}{2} U_4 = \left(\frac{5}{2} b^4 + 10b^3 + 21b^2 + 25b + 12\right) e^{-2b},$$

in which we can easily trace the following law:

$$V_n - \frac{1}{2} U_n = \frac{1}{2}\left[\frac{(n+1)!}{n!} b^n + \frac{1!(n+2)!}{3!(n-1)!} b^{n-1} + \frac{2!(n+3)!}{5!(n-2)!} b^{n-2} + \ldots + n!\right] e^{-2b} \ldots (10)$$

Out of equation (8) and this one it is evident that $V_{n+1} - \frac{1}{2} U_{n+1}$ follows the same law; so the relation (10) is proved.

Mathematics. — *"An article on the knowledge of the tetrahedral complex."* By Dr. Z. P. BOUMAN. (Communicated by Prof. JAN DE VRIES).

§ 1. When for an arbitrary ray out of a tetrahedral complex P_i represents the point of intersection with the face $A_k A_l A_m$ of the tetrahedron, then

$$R\,p_2\,p_5 + p_3\,p_6 = 0,$$

where R represents the given anharmonic ratio of the complex and p_i $(i = 1 \ldots 6)$ are the PLÜCKER coordinates of lines.

By using the condition necessary for each ray of the complex, namely

$$p_1\,p_4 + p_2\,p_5 + p_3\,p_6 = 0$$

the equation of the complex becomes

$$A\,p_1\,p_4 + B\,p_2\,p_5 + C\,p_3\,p_6 = 0,$$

where the anharmonic ratio is given by

$$R = \frac{B-A}{C-A}.$$

A given tetrahedral complex can always transform itself projectively into another one with the same anharmonic ratio in regard to the faces of the rectangular system of coordinates and the plane at infinity.

§ 2. After having executed this transformation we can examine whether a surface with two independent parameters can be found in such a manner that the normals to be erected in an arbitrary point on the ∞^1 number of surfaces passing through that point, are rays of the given tetrahedral complex.

To this end we make the two determining points to lie infinitely close to each other on each ray of the complex, so that each ray is determined by one point (x, y, z) and the direction (dx, dy, dz) in that point. The coordinates of lines now take the form:

$$p_1 = x\,dy - y\,dx, \quad p_2 = y\,dz - z\,dy, \quad p_3 = z\,dx - x\,dz,$$
$$p_4 = -\,dz, \qquad p_5 = -\,dx, \qquad p_6 = -\,dy.$$

So the equation for the complex becomes:

$$A\,(x\,dy - y\,dx)\,dz + B\,(y\,dz - z\,dy)\,dx + C\,(z\,dx - x\,dz)\,dy = 0.$$

If now every ray of the complex is to be at right angles to a surface $z = f(x, y)$, then we have for each ray in each point of the surface:

$$dx : dy : dz = p : q : -1 ,$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

So the differential equation of the surface becomes:

$$-pqz(B-C) + yp(A-B) + xq(C-A) = 0$$

or

$$z - \frac{x}{p}\frac{1}{R-1} - \frac{y}{q}\frac{R}{1-R} = 0 .$$

The complete integral with two parameters C and C_1 becomes:

$$z = \pm\sqrt{\frac{1}{R-1}}\sqrt{x^2 - C} \pm \sqrt{\frac{R}{1-R}}\sqrt{y^2 - C_1} .$$

It represents a surface of order four.

It is evident out of the equation that for $R = \frac{1}{R'}$ the surface remains the same; only the X- and the Y-axes have been interchanged. (This is geometrically immediately made clear). So we have but to examine the surface for, let us say, $R > 1$.

§ 3. It must be possible to find the equation of the cone of the complex in a definite point out of the equation of the surface because that cone is the locus of the normals to the ∞^1 number of surfaces, passing through the point under consideration. If α, β, γ represent the cosines of direction of a ray of the complex in the point x_1, y_1, z_1 then

$$p = -\frac{\alpha}{\gamma}, q = -\frac{\beta}{\gamma} .$$

Substituting this in the differential equation and eliminating α and β by means of the equations of the ray of the complex, namely

$$\frac{x-x_1}{\alpha} = \frac{y-y_1}{\beta} = \frac{z-z_1}{\gamma} ,$$

we find for the cone of the complex:

$$(R-1)\, z_1 (x-x_1)(y-y_1) - R y_1 (x-x_1)(z-z_1) + x_1 (y-y_1)(z-z_1) = 0.$$

The planes of the coordinates forming the singular surface of the complex, the cone of the complex must degenerate for each point of one of these planes. For the point $P(x_1, y_1 = 0, z_1)$ the cone breaks up into $y = 0$ and into $x_1 z + (R-1) z_1 x = R z_1 x_1$, i. e. a plane passing through P and parallel to the Y-axis. This plane is at right angles to OP, if this line has for equation $z = \pm x\sqrt{\frac{1}{R-1}}$. (Comp. § 4).

§ 4. The drawing of the surfaces to be found offers no difficulties. For $R>1$ (§ 2) we must take C_1 positive and then we have to distinguish the cases $C \gtreqless 0$.

So for $C>0$ the surface consists of two separated parts connected by points forming parts of a double conic in the XOY-plane. The planes $x=\pm\sqrt{C}$ touch both parts according to equal ellipses and no points lie between with $z>0$.

The section with the XOZ-plane consists of two hyperbolae with centres $\left(z=\pm\sqrt{\frac{RC_1}{R-1}}\right)$ on the Z-axis. At infinity they are connected twice, and intersect each other in the points of intersection of the double conic with the X-axis. The hyperbolae coincide in the planes $y=\pm\sqrt{C_1}$, where the common vertex of the double conic is lying.

C becoming smaller, the two parts of the surface approach each other and for $C=0$ the conics meet in the planes $x=\pm\sqrt{C}$. The surface becomes a ruled surface, so it breaks up into two cylinders with axes in the XOZ-plane.

The axes have for equation $z=\pm x\sqrt{\frac{1}{R-1}}$. (Comp. § 3). The section perpendicular to these axes is a circle which is in accordance with the signification of the axes as found in § 3.

§ 5. It is known that the normals of a system of similar, concentric ellipsoids form a tetrahedral complex [1]). So this system must be a particular integral of the above-mentioned differential equation.

Let us put $C=gC_1+h$ (g and h being constants) and let us operate in the ordinary way; we find C and C_1 as functions of the variables out of:

$$y^2-C_1=R\frac{gy^2+h-x^2}{g(g+R)},$$

$$x^2-C=-g\frac{gy^2+h-x^2}{g+R}.$$

Substitution in the complete integral furnishes:

$$z^2\frac{g(1-R)}{g+R}-gy^2+x^2=h.$$

Let us put in this equation $g=-\frac{a^2}{b^2}$, and let c be the axis along the Z-axis; we shall then find if we take a^2 positively

[1]) Dr. J. DE VRIES: On a special tetrahedal complex. Proceedings of Febr. 25 1905, Vol. XIII, pages 572—577.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=h', \text{ with } R=\frac{a^2-c^2}{b^2-c^2}.$$

Likewise $\left(g=-\frac{a^2}{b^2},\ a^2 \text{ negative}\right)$ the system of hyperboloids with two sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=h', \text{ with } R=\frac{a^2+c^2}{b^2+c^2}$$

and also $\left(g=\frac{a^2}{b^2},\ a^2 \text{ positive}\right)$ the system of hyperboloids with one sheet

$$\frac{z^2}{c^2}+\frac{x^2}{a^2}-\frac{y^2}{b^2}=h', \text{ with } R=\frac{c^2-a^2}{c^2+b^2}.$$

§ 6. The "curves of a complex" are curves whose tangents are rays of the complex. The coefficients of direction (α, β, γ) in a definite point (x, y, z) must therefore be proportional to

$$p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, -1,$$

of one of those surfaces through that point. From this ensues that $p=-\frac{\alpha}{\gamma}$ and $q=-\frac{\beta}{\gamma}$, whilst x, y, z, p and q must satisfy the equation:

$$z-\frac{x}{p}\frac{1}{R-1}-\frac{y}{q}\frac{R}{1-R}=0.$$

So the quantities $x, y, z, \alpha, \beta, \gamma$ must satisfy:

$$\frac{z}{\gamma}-\frac{y}{\beta}\frac{R}{R-1}+\frac{x}{\alpha}\frac{1}{R-1}=0.$$

Let a curve of the complex be given by:

$$x=f_1(s),\quad y=f_2(s),\quad z=f_3(s),$$

where s need not of necessity represent the length of the arc, then:

$$\frac{f_3(s)}{f_3'(s)}+\frac{f_2(s)}{f_2'(s)}\frac{R}{1-R}+\frac{f_1(s)}{f_1'(s)}\frac{1}{R-1}=0.$$

Amongst others all curves for all values of p to be represented by

$$x=\lambda(l+s)^p,\quad y=\mu(m+s)^p,\quad z=\nu(n+s)^p$$

satisfy this equation if only

$$\frac{l-n}{m-n}=R,$$

which condition can be satisfied by putting $l=B$, $m=C$, $n=A$.

For $p = -1$ these are twisted cubics. If we bring these through a point (x_1, y_1, z_1) the ∞^1 curves all lie on the cone of the complex of this point. This holding for each point, the bisecants (and not only the tangents) are rays of the complex.

Indeed, all the twisted cubics pass through the vertices of our tetrahedron and the four planes passing through a bisecant and these four points have thus a constant anharmonic ratio. From this ensues that the bisecants intersect the four planes of coordinates in the same anharmonic ratio.

For $p = 1$ we have the rays of the complex themselves.

For $p = 2$ we have conics which can be nothing but conics of the complex, e.g. for $s = -l$ the curve touches the plane YOZ, etc.

For $p = 3$ we have twisted cubics whose bisecants are not rays of the complex, etc.

In general the tangents to the "curves of a complex" lie always in linear congruences belonging to the tetrahedral complex. For such a tangent namely we have

$$(l + s)\frac{dx}{x} = (m + s)\frac{dy}{y} = (n + s)\frac{dz}{z}.$$

From this ensues among others:

$$(n + s)\frac{dz}{z} = \frac{(l + s)\,dx + k\,(m + s)\,dx}{x + ky}. \quad (k \text{ an arbitrary constant.})$$

This is evidently always satisfied by rays of the complex, satisfying at the same time:

$$x dz - z dx = k\,(z dy - y dz) \quad \text{and} \quad k dy = -R dx,$$

for which we can write in coordinates of lines:

$$p_3 = k p_2 \quad \text{en} \quad -k p_6 = R p_5.$$

These satisfy the equations of the tetrahedral complex and lie in congruences; the two linear complexes determining such a congruence, are themselves special, and the position of their axes is evident from their equation.

§ 7. Finally it proves to be simple to bring in equation the curves which are drawn on an arbitrary surface in such a way that the cone of the complex touches the surface in each point of the curve.

Let the surface be $f(x, y, z) = 0$ and the ray of the complex $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$, passing through the point x_1, y_1, z_1 of the surface.

A ray of the complex in the tangential plane must satisfy

$$\alpha \frac{\partial f}{\partial x_1} + \beta \frac{\partial f}{\partial y_1} + \gamma \frac{\partial f}{\partial z_1} = 0,$$

and further according to the differential equation

$$(R - 1)\, z_1\, \alpha\beta - R\, y_1\, \alpha\gamma + x_1\, \beta\gamma \text{ must be equal to } 0.$$

The two rays of the complex in the tangential plane have but to be made to coincide. The condition is:

$$-4\, R(R-1)\, z_1\, y_1\, f_3\, f_3 = [-(R-1)\, z_1\, f_3 + R\, y_1\, f_2 + f_1\, x_1]^2,$$

where f_1, f_2, f_3 represent the differential quotients of f according to x, y and z respectively, whilst analogous relations are easy to deduce.

From this ensues that the required curve is the intersection of

$$f(x, y, z) = 0$$

and

$$-4\, R(R-1)\, z\, y\, f_2\, f_3 = [-(R-1)\, z\, f_3 + R\, y\, f_2 + f_1\, x]^2.$$

Without entering into further details I only wish to observe that when $f(x, y, z) = 0$ represents a plane, the curve can be nothing but the conic of the complex. From the above mentioned equations we therefore find a parabola (the conic of the complex touches the tetrahedron plane at infinity) touching the three planes of coordinates of the rectangular system of axes.

Physiology. — "*On the excretion of creatinin in man*". By C. A. PEKELHARING. Report of a research made by C. J. C. VAN HOOGENHUYZE and H. VERPLOEGH.

As the muscle tissue in herbivora as well as in carnivora always contains a not unimportant amount of creatin, and creatinin is daily excreted with the urine it may be concluded, that creatin is formed as a product of metabolism in the muscles, and having entered the blood is at least for a part excreted by the kidneys in the form of the anhydride, creatinin.

But no agreement has been obtained about the question whether the forming of creatin is bound to the labour, the contracting of the muscles. To answer that question, researches have been made whether the amount of creatinin excreted by the kidneys augments after muscular labour. Different investigators have obtained different results. VAN HOOGENHUYZE and VERPLOEGH have resumed the research anew, using a new method to determinate the amount of creatinin in the urine, which was published some time ago by FOLIN [1]). The

[1]) Zeitschr. f. Physiol. Chemie, Bd. XLI, S. 223.

method of FOLIN is founded on the reaction of JAFFÉ, which consists in adding picric acid and an excess of caustic soda to a solution of creatinin, whereat the liquid takes a brown colour, which cannot be discerned from the colour of a solution of bichromate of potassium. This reaction is employed in the following way: 5 cc. of urine is mixed with 15 cc. picric acid 1,2 % and 5 cc. of caustic soda 10 %. After 5 minutes water is added to a volume of 250 cc. This solution is compared by FOLIN in the colorimeter of DUBOSCQ with a $^1/_2$ normal solution of bichromate of potassium of which a column 8 mm. high shows exactly the same intensity of colour as a column 8,1 mm. high of a solution of 10 mgr. creatinin with 15 cc. picric acid solution and 5 cc. caustic soda diluted to 500 cc. Instead of the colorimeter of DUBOSCQ, VAN HOOGENHUYZE and VERPLOEGH used a little instrument, constructed after their indication, which answered completely to their demands. Immediately after each determination each of them performed 5 readings of the height of the solution of creatinin at which its colour had just the same intensity as a column 8 mm. high of the solution of bichromate of potassium. The several readings of which the average was taken, never differed more than 0,2, only very seldom more than 0,1 mm.

It proved meanwhile that the temperature has influence on the reaction in that sense that the colour of the creatinin solution becomes deeper by increase of temperature. Therefore the water used for the diluting was always kept at a temperature scarcely differing from 15° C. The relation found by FOLIN was affirmed. A solution of 10 mgr. of pure creatinin in 500 c.c. treated in the indicated way produced as the average of 10 readings 8.14 m.m. (max: 8.2, min. 8.1) out of which a quantity of 9.951 instead of 10 mgr. would be deduced.

The results become less exact when the concentration of creatinin is much larger or smaller than 10 mgr. in 500 c.c. Therefore the determination was repeated when the readings became higher than 10.5 or lower than 5, with 10 c.c. urine, in the first case diluted to 250 in the second to 1000 c.c. The method of FOLIN had great advantages over the method of NEUBAUER used till now, in which the creatinin is precipitated out of an alcoholic extract of the urine by means of chloride of zinc and after that weighed. Not only that the method of FOLIN takes much smaller quantities of urine, so that it renders it easy to discern by the examination of different portions of urine the oscillations in the secretion in the course of the day, but it is also more reliable. With the method of NEUBAUER there is always some danger that under the influence of the alkaline reaction arising from the addi-

tion of milk of lime to separate the phosphates, a part of the creatinin is changed into creatin. This danger may be lessened but not wholly avoided by acidifying the filtrate before evaporation by means of hydrochloric acid, after which at the end the hydrochloric acid must be eliminated by addition of sodium acetate in order not to hinder the precipitation of creatinin zinc chloride. But there are other difficulties connected with the method of NEUBAUER which can never be totally removed. The urine is after the removal of the phosphates concentrated till it obtains the consistency of syrup and is than extracted with alcohol. In the mass of salts rendered hard by the contact with alcohol, a part of the creatinin may be retained undissolved. If in order to eliminate this difficulty the urine is not very much evaporated, there arises another source of error. The alcohol is diluted by the still resting water and the consequence is that now the creatinin-zincchloride crystallises only partially. For this compound is insoluble in absolute alcohol but not in alcohol containing water. A too small quantity of creatinin is therefore always found by the application of this method.

VAN HOOGENHUYZEN and VERPLOEGH have investigated the solubility of creatinin-zincchloride in alcohol by putting dried crystals, prepared from urine and purified as much as possible, in closed bottles under alcohol of different strength at the temperature of the room under repeated shaking and by determinating afterwards, by means of FOLIN's method, how much creatinin was dissolved in the alcohol. They found:

in 100 C.C. alcohol	99 %	trace of	creatinin.
„ „ „ „	93 %	5.6 mgr.	„
„ „ „ „	72 %	32.1 „	„
„ „ „ „	50 %	104.5 „	„

In connection with this they obtained out of urine more creatinin-zincchloride when the alcoholic extract before the addition of chlorid of zinc was again evaporated to almost dryness and then dissolved by strong alcohol, than with the usual method. They could still show creatinin in the liquid filtered off from the creatinin-zinc-chlorid as well by the reaction of WEYL as by that of JAFFÉ. So the method of NEUBAUER always gives a loss of which the amount cannot be estimated. One is therefore not entitled to attribute much value to the little oscillations in the output of creatinin found by applying this method.

By the method of FOLIN on the contrary such a source of uncertainty does not exist, when the time of the reaction — 5 minutes — is rightly observed, the liquid is brought to the exact volume with

water of the temperature of the room and when the determination is completed immediately afterwards.

VAN HOOGENHUYZE and VERPLOEGH have investigated by themselves whether increase of the secretion of creatinin in consequence of muscular labour could be observed. For that purpose in every series of experiments the urine was collected every day at appointed times namely in the morning, in the first series of experiments at 9, in the following at 8 o'clock, in the afternoon at 12 o'clock and at $4^1/_2$ o'clock, at night at $11^1/_2$ o'clock. Every portion was measured and divided into two equal halves. One of the halves was used for an estimation of creatinin, the other halves were mixed, after which the quantity of creatinin in the mixture was determined and moreover an estimation of nitrogen was performed after the method of KJELDAHL.

In this way the determination of creatinin was also controlled. In all the series of experiments the conformity of the figure of the total quantity of creatinin and the sum of the four portions was very gratifying. The quantity of urine of one day was that collected from 12 o'clock in the afternoon till the following morning 8 or 9 o'clock.

During each series of experiments a fixed amount of food was taken, every day the same. Only in the first series coffee and tea were still taken, in the later series only water.

I. From April the 8th—24th 1904, seventeen days at a stretch, food was taken which consisted of bread, butter, cheese, milk, oatmeal, sugar, meat, eggs, potatoes and rice, daily an equal portion of each.

The food contained:

for v. H. 118 gr. proteid 146 gr. fat, 326 gr. carbohydrat.; 40,8 Cal. p. Kg.
„ V. 115 „ „ 81 „ „ 327 „ „ 38.6 „ „ „

On working days moreover both consumed 50 gr. sugar.

The 11th, the 16th and the 21th of April bicycle excursions were undertaken at which they rode steadily on for $2^1/_2$ à 3 hours without resting. The other days were spent in the laboratory while the evenings were passed peacefully.

The excretion of creatinin underwent no perceptible change in consequence of the muscular labour. With both investigators it oscillated not unimportantly during the whole experiment. It amounted on an average to:

v. H. 14 days of rest 2.116 gr. daily (max. 2.401, min. 1.821 gr.)
V. „ „ „ „ 1.998 „ „ („ 2.158 „ 1.858 „)
v. H. 3 workingdays 2.147 „ „ („ 2.325 „ 1.925 „)
V. „ „ 2.015 „ „ („ 2.058 „ 1.949 „)

The difference is so small that no value must be attached to it. On the days which followed on the muscular exertion the figures of the creatinin remained within the usual daily oscillations.

The secretion of nitrogen was rather irregular with both during the whole experiment.

From June the 22nd till July the 2nd 1904 (eleven days) the experiment was repeated with less food which in particular was less rich in proteid. It contained:

for v. H. 71.5 gr. proteid, 125 gr. fat, 351 gr. carbohydr.; 33.7 Cal. p. Kg.
„ V. 80.5 „ „ 74.75 „ „ 358 „ „ 34.6 „ „ „

On July the 1st a bicycle excursion of three hours was undertaken (50 KM.).

The excretion of creatinin amounted on an average to:

	10 days of rest	Workingday
v. H.	1.983 (max. 2.042, min. 1.809 gr.)	1.997 gr.
V.	2.039 („ 2.174, „ 1.920 „)	2.049 „

On the days which followed the day of muscular labour the excretion of creatinin did not increase either.

III. Whereas till now meat was still taken, in the series of experiment II daily 50 gr., the experiment was now taken with food which contained no creatinin at all, moreover it was made poorer in proteid. The experiment lasted from July the 7th till the 29th 1904, 23 days at a stretch.

From July the 7th till 18th only bread, butter, cheese, rice and sugar were taken containing:

for v. H. 50 gr. proteid, 115 gr. fat, 344 gr. carbohydr.; 31.2 Cal. p. Kg.
„ V. 50 „ „ 74 „ „ 344 „ „ 33.8 „ „ „

From July the 18th rice was partly replaced by potatoes and the quantity of butter was decreased so that the ration became:

for v. H. 47 gr. proteid, 98 gr. fat, 337 gr. carbohydr.; 29,5 Cal. p. Kg.
„ V. 47 „ „ 64 „ „ 337 „ „ 30.1 „ „ „

On July the 28th and the 29th 5 eggs were daily added to this food.

On July the 15th, the 20th and the 23rd muscular labour was again performed while the other days were passed in the laboratory with occupations which exacted only little exertion of the muscles. On July the 15th a bicycle excursion was undertaken in which 54 K.M. were covered in three hours. On July the 20th and 23rd fatiguing indoors gymnastics were performed for $2^1/_2$ hours at a stretch with halters of 10 K.G. and with the chest-expander and

the combined developer of SANDOW; care was taken that all the muscles of the body and the extremities were used.

When the first three days of the scanty diet, July the 7th, 8th and 9th, in which the secretion of nitrogen fell with v. H. from 14,562 to 9,045 gr. and with V. from 13,721 to 10,234 gr. are not counted, as belonging to a transition-period, and neither the last two days, July the 28th and 29th at which about 30 gr. more proteids daily were taken, it appears that the excretion of creatinin has amounted in 15 days of rest on an average every day to:

v. H. 1.836 gr. (max. 1.935, min. 1.693 gr.)
V. 1.962 „ („ 2.079, „ 1.876 „)

while on the working days was found:

v. H. July the 15th 1.908, July the 20th 1.921 and July the 23th 1.974 gr. creatinin.
V. July the 15th 2.142, July the 20th 1.947, and July the 23rd 1.937 gr. creatinin.

Here then the figure with v. H. is always, with V. once above the average on the working day. Meanwhile the deviations do not surpass the oscillations, which are always found, also without important exertions of the muscles.

The figure found with V. on July the 15th does, it is true, surpass the maximum in the period of the days of rest, but the difference 0,063 gr. is so slight, that no value must be attached to that, in connection to the lower figures of the two other workingdays.

On the two last days of the series on which no muscular labour was performed, but on which more proteid was taken, the excretion of creatinin was:

v. H. July the 28th 1.955 gr., and July the 29th 1.959 gr.
V. „ „ „ 2.053 „ „ „ „ „ 1.984 „

while with both the secretion of nitrogen increased from about 8 gr. to 11 gr. daily.

IV. In September 1905 a new experiment was taken, to examine firstly whether preceding muscular exercise might perhaps bring some change in the result, secondly to investigate the influence of excessive labour and thirdly to see whether the excretion of creatinin would be increased with excessive labour and totally insufficient food.

After performing daily for three weeks at a stretch indoor gymnastics after the method of SANDOW, the experiment was begun September the 26th with food of the same composition as was used July the 18th till the 27th, hardly sufficient and poor in proteid. This food was

taken nine days at a stretch till Oct. the 4th. On September the 29th exercises were performed with SANDOW's implements for $2^1/_2$ hours with short intervals. On October the 2nd excessive labour was done, consisting of a walk of 21 KM. in the morning from 9 till 12 o'clock, a walk of 10 K.M. in two hours in the afternoon and working with halters for $^1/_2$ hour in the evening. On the six days of rest between September the 27th and Oct. the 4th (on the first day Sept. the 26th the urine was not examined) there was excreted on the average every day:

v. H. 1.859 (max. 1.977, min. 1.755) gr. creatinin
V. 1.925 („ 2.047, „ 1.860) „ „

while on Sept. the 29th there was found:

v. H. 2,001 V. 1.979, gr. creatinin
On October 2nd „ 1.859 „ 1,945, „ „

That not too much importance for the influence of muscular labour on the secretion of creatinin must be attached to the somewhat high figure of v. H. on September the 29th becomes clear when the separate portions of that day are considered. In the first portion of that day, that is in the urine excreted in the morning between 8 and 12 o'clock, so before muscular exertion was begun, 0,404 gr. creatinin was already found to 0,331 gr. and 0,345 gr. in the corresponding portions of the preceding and the following day.

After ordinary food had been taken for nine days, food was taken in absolutely insufficient quantity for five days at a stretch, consisting of bread, potatoes, butter and cheese. It contained:

for v. H. 36.6 gr. proteid 43 gr. fat 186 gr. carbohydrate; 15 Cal p. Kg.
„ V. 29.7 „ „ 34.7 „ „ 158 „ „ ; 15 „ „ „

On Oct. the 16th a bicycle ride of 42 K.M. in $2^1/_2$ hours was undertaken in the morning. In the first hour 20 K.M. were done but after that they could progress but slowly from hunger and fatigue.

In the afternoon a walk of 16 K.M. was taken from 2 till 5 o'clock and afterwards in the evening they worked with halters. The result was that both felt still very tired the next day.

The calculation of the average has no value in this short experiment. The course of the excretion of the creatinin was as followed:

	v. H.	V.	
Oct. the 14th	2.020	1.908	
„ „ 15th	1.702	1.934	
„ „ 16th	1.775	1.899	workingday
„ „ 17th	1.831	1.938	
„ „ 18 h	1.861	1.868	

Here too where the food was not sufficient for the organism to defray the costs of the muscular labour, as appeared also from the increase of the nitrogen secretion on the workingday, one can certainly not speak of distinct influence of muscular labour on the excretion of creatinin.

It is however different when no food is taken at all for days.

VAN HOOGENHUYZE and VERPLOEGH had an opportunity to make observations about this too on the "Hungerkünstlerin" FLORA TOSCA a strong, young woman, who lent herself for the investigation during a starving period at the Hague, in a room which was opened to the public night and day. The urine was collected every day in three portions, in the morning from 10 o'clock till 4 o'clock in the afternoon, from 4 o'clock in the afternoon till 10 o'clock in the evening and from 10 o'clock in the evening till 10 o'clock the next morning; it was sent every day at a fixed time to the Physiological Laboratory in Utrecht and was there examined at once.

In the morning of June the 10th 1905 the last food was taken; after that nothing but mineral water (Drachenquelle) till June the 25th. Besides creatinin several other constituents of the urine were determinated daily; about this it will be sufficient to mention that from the course of the secretion of nitrogen, urea, uric acid and phosphoric acid it appeared sufficiently that no food was taken.

During the whole hunger-period of fourteen days complete bodily rest was observed as much as possible save on June the 17th when TOSCA during two hours with short rests, under direction of VERPLOEGH, was occupied with gymnastic exercises with halters of 1 KGr. 13 different movements were made, the first ten 20 times each, the last three 10 times each. The movements were so chosen that as many muscles as possible were set to work.

The examination of the urine showed now that in hungering the secretion of the creatinin as well as of the other products of metabolism steadily decreased. But the muscular labour suddenly produced an undeniable increase, not on the same day, but on the following. Still on the third day the influence was to be perceived, which however also was the case with connection to the total quantity of nitrogen. On the first day, when food was still taken, the quantity of creatinin amounted to 1,087 gr. Later on it decreased rapidly and rather regularly till on the 8th day. On June the 17th, the day of the muscular labour, it amounted only to 0,469 to rise the following day to 0,689. In the three days before the muscular labour 1,662 was secreted, in the three following days 2,006 gr. creatinin. After that the secretion decreased almost to 0,5 gr, daily, to remain rather constant then.

From the above mentioned it appears that even with perfectly regular food and with avoiding of all excessive muscular labour the daily secretion of creatinin, as was communicated already in 1869 by K. B. HOFMANN [1]), undergoes rather important oscillations. This is not sufficiently taken into consideration by those authors who as MOITESSIER [2]) and as GREGOR [3]) have deduced from their results with series of experiments of three, four or five days, where the creatinin was precipitated from the alcoholic extract of the urine as a compound of zincchloride, that the excretion of creatinin increased as a result of muscular labour. It seems therefore to me that more value may be attached to the conclusion, which v. HOOGENHUYZE and VERPLOEGH drew from their observations, that in man only then increase of excretion of creatinin is caused by muscular labour when the organism is forced, by abstaining from food, to live at its own costs.

If the creatinin which is found in the urine of normal and normally fed men and animals is not to be considered, even were it for a small portion, as a product set free by the contraction of the muscle fibre, the question arises what signification must be given to this constituent of the urine.

Since MEISSNER's researches [4]) it is known that to make use of meat as a food must lead to the excretion of creatinin, as creatin and creatinin, brought into the blood either by resorption out of the intestinal canal or by injection under the skin completely or almost completely is excreted as creatinin by the kidneys.

The quantity of creatin in meat is rather important. It is usually mentioned as 0,2 à 0.3% of the fresh muscle substance [5]). With the aid of FOLIN's method v. H. and V. have determined the amount of creatin in muscle. 500 gr. meat freed as carefully as possible of fat and tendons and minced was mixed with chloroform water and was pressed out after standing for some hours at the temperature of the room. This was repeated twice. After that the pressed out meat was boiled for two hours with water and after cooling pressed out anew. The filtrates were mixed, boiled at weak acid reaction to remove proteids, after cooling filled up to 4000 c.c, and then filtered. 500 c.c. of the filtrate was concentrated to 100 c.c. and filtered anew, 80 c.c. of this filtrate was boiled with 50 c.c. *n*

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1) Virchow's Archiv. Bd. XLVIII S. 358.

2) Thèse Montpellier 1891.

3) Zeitschrift f. Physiol Chemie. Bd. XXXI S. 98.

4) Zeitschr. f. rat. Med. Bd. XXXI, 1868, S. 234.

5) VOIT. Zeitschr. f. Biol. Bd. IV, 1868. S. 77.

H_2SO_4 48 hours in the waterbath, to change all creatin into creatinin. After that the quantity of creatinin was determined colorimetrically. Every time a determination of the same kind of meat of different animals was made twice. So the following was found:

Beef	I	3.688	gr.	creatinin:	4.378	gr.	creatin	p. Kg.	meat
	II	3.898	„	„	4.522	„	„	„ „	„
Mutton	I	3.499	„	„	4.059	„	„	„ „	„
	II	3.608	„	„	4.185	„	„	„ „	„
Pork	I	3.718	„	„	4.313	„	„	„ „	„
	II	4.070	„	„	4.721	„	„	„ „	„
Horse	I	3.244	„	„	3.763	„	„	„ „	„
	II	3.395	„	„	3.948	„	„	„ „	„

Even with an abundant use of meat or beef-tea the creatinin excreted by the kidneys (1,5 à 2 gr. or still more in 24 hours) can but for a part be derived from the food. It is moreover well-known and by the above mentioned researches proved anew that the secretion of creatinin sinks not or scarcely under the norm, when the food does not at all contain creatin or creatinin. The organism itself forms creatin as a product of metabolism from the proteids. It would be possible that the nature of the proteid taken up as food was of signification for the forming of creatin. In that case it was possible that especially such proteids would produce creatin, out of which by hydrolysis much arginin, a more complicated derivative of guanidin could be obtained. According to the researches of KOSSEL and his disciples, from gelatin twice as much arginin can be obtained as from casein; out of gelatin 9.3 %[1]), out of casein 4.8 %[2]). VAN HOOGENHUYZE and VERPLOEGH have therefore examined by a new series of experiments whether the use of casein or gelatin increases the secretion of creatinin and if such is the case in what measure.

V. On April the 7th 1905 a beginning was made with the use of the same food as in series IV.

v. H.	47	gr.	proteid,	98	gr.	fat,	337	gr.	carbohydr.;	29.5	Cal. p. Kg.
V.	47	„	„	64	„	„	337	„	„	30	„ „ „

On April the 12th, 13th, 14th 50 gram casein was taken prepared after HAMMARSTEN from cow-milk, in the afternoon at 12 o'clock 25 gr. and in the evening at 6 o'clock once more 25 gr. To leave the total chemical energy of the food unchanged, so much fewer potatoes were taken on these days that the quantity of carbohydrates fell from 337 to 287. After that the food was taken as on April the

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[1]) Zeitschrift f. Physiol. Chem. Bd. XXXI, S. 207.

[2]) Ibid. Bd. XXXIII, S. 356.

7th till April the 19th. April the 20th 21st and 22nd 50 gram commercial gelatin, well washed in water, was taken every time in two portions, each of 25 gr., just as the casein instead of 50 gr. carbohydrate. On April 23rd and 24th the first diet was again taken.

In 10 days in which the food daily taken contained 47 gr. proteid (the first two days April the 7th and the 8th, which were still under the influence of the food taken the preceding days, the urine was not examined) the secretion of creatinin amounted to:

v. H. on the average 1.813 gr. (max. 1.921 min. 1.706 gr.) daily
V. „ „ „ 1.850 „ („ 1.990 „ 1.723 „) „

On the days on which casein or gelatin was taken the secretion of nitrogen increased but the secretion of creatinin not or scarcely. It amounted on the three casein-days to:

v. H. on an average 1.913 gr. (max. 2.009, min. 1.836 gr.) daily
V. „ „ „ 1.897 „ („ 1.934, „ 1.834 „) „

and on the three gelatin-days:

v. H. on an average 1.800 gr. (max. 1.813, min. 1.783 gr.) daily
V. „ „ „ 1.872 „ („ 1.811, „ 1.868 „) „

Just as in the series of experiments III as was mentioned above, where, after the daily addition of 5 eggs to food which contained 47 gr. proteid, only a too insignificant increase of the secretion of creatinin was found to attach any value to it, it appeared now that the addition of casein and gelatin had no important influence whatever, although the added proteid was daily resorbed and desintegrated in the body, as the determination of nitrogen taught.

A short time ago Folin has communicated ample researches about the constituents of human urine and has come to conclusions [1]) with which the observations of van Hoogenhuyze and Verploegh are quite in accordance.

In 1868 Meissner has drawn the conclusion from his observations that the origin of creatinin in the organism of mammals must be quite different from that of the urea with which most of the nitrogen is excreted from the body [2]). Folin draws this conclusion anew and, in connection with his observations about the secretion of other nitrogen containing substances and sulphur-compounds, starts from this point in proposing a new theory about the desintegration of proteid in the animal body, which he puts in the place of the wellknown theories of Voit and of Pflüger. In considering the desintegration of proteids in the body, there has been, argues Folin, generally laid

[1]) Amer. Journ. of Physiol. Vol. XIII, p 45. p. 66 p. 117.
[2]) l. c. S. 295.

stress almost only on the total quantity of nitrogen excreted, in relation to the quantity taken up in the food, and not enough attention has been paid to the quantities of each of the different nitrogenous products of metabolism which are excreted with the urine.

When the quantity of proteid in the food is enlarged or diminished then the secretion of nitrogen increases or decreases till after a short time a condition of equilibrium has been again obtained when intake and output of nitrogen are alike. The variability of the metabolism of proteids does not manifest itself in connection with all nitrogenous substances but for the greater part with connection to the urea. The secretion of creatinin on the contrary and also in a less degree that of uric acid is apparently independent of the richness of the food in proteid. We must distinguish a desintegration of proteid variable under the influence of the food, on which depends in the first place the forming of urea and which according to FOLIN's conception takes place for the greater part if not wholly in the digestive organs — in the cavity and in the mucosa of the intestine and in the liver — and beside a much less variable desintegration of proteid in the different organs which does not immediately depend on the food but on the function of the tissues. In the tissues there arise undoubtedly nitrogenous products of desintegrating of different composition. To them belongs as has been stated by NENCKI, SALASKIN and their collaborators ammonia, which is changed into the harmless urea by the liver. Moreover urea is formed in the organism in other places than the liver. This product of metabolism proceeds thus for a part, as FOLIN expresses it "endogenously" in consequence of the rather regular metabolism of proteid in the tissues and for another part "exogenously" in larger or smaller quantities, as more or less proteid is taken up in the digestive canal. It is however not possible to distinguish these two parts from each other in the urine.

But on the contrary the secretion of creatinin, on which the digestion of the food when it contains no creatin has no direct influence, gives an indication about the intensity of the desintegration of proteid in the tissues. In this respect the muscular tissue, must be thought of in the first place, but not exclusively, as creatin is formed undoubtedly in other tissues too.

It does not seem necessary to accept that all the creatin which is formed in the tissues is excreted as creatinin. The observations of MEISSNER give already rise to the supposition that creatin must be considered as an "intermediate" product of metabolism, as has been stated by BURIAN and SCHUR for the uric acid. MEISSNER at least could not quite retrace in the urine the creatinin brought into the

circulation. He did find, it is true, that after injection of creatin under the skin, not only the whole injected quantity was excreted again with the urine, but also 20 mgr. creatinin with it, but it remained uncertain how much of it proceeded from the metabolism of the animal itself.

To obtain an insight into this v. H. and V. have made anew an experiment in which the same food was taken, with 47 gr. proteid daily, as in the preceding experiment.

VI. The experiment lasted from Aug. the 17th till the 28th 1905. On the first day the urine was not examined. The oscillations in the secretion of creatinin were very insignificant. In five days from Aug. the 18th till the 22nd there was secreted:

v. H. average 2.023 gr. (max. 2.029, min. 2.017 gr.) daily
V. „ 2.028 „ („ 2.029, „ 1.930 „) „

On Aug. the 23rd each of them took in one portion 500 mgr. pure creatinin dissolved in water. On the same day there was excreted:

v. H. 2.420 gr. and V. 2.508 gr. The next day:
V. 2.030 „ „ „ 2.073 „ „ „

On Aug. the 26th each of them took again 500 mgr. creatinin but divided into 10 portions, 50 mgr. every hour. Now also the creatinin was found back the same day for the greater part in the urine. The excretion amounted to:

v. H. Aug. the 25th 1.998 Aug. the 26th **2.425** Aug. the 27th 1.940 Aug. the 28th 1.951 gr.
V. Aug. the 25th 2.045 Aug. the 26th **2.467** Aug. the 27th 2.035 Aug. the 28th 1.968 gr.

At least in three of the four determinations a part of the creatinin brought into the blood was not found back in the urine.

From this experiment, which has still to be completed with others, in which creatin will be taken instead of creatinin, it appears how distinctly every change of some importance in the excretion of creatinin can be shown with the aid of FOLIN's method. So it gives the more reason to trust the results of the above mentioned series of experiments, and the conclusion derived from them, that creatin is a product of metabolism which is not formed at the contraction of the muscle-fibre, but proceeds in muscles and other organs by the desintegration of proteid to which is bound the life of the cells, without regard to the developing of energy to which they are able in performing their peculiar functions. Only then when the organism is deprived of food and must therefore seek the power of performing

labour in itself, the material which the muscles want for contraction is taken from the proteids of the tissue; for this the tissues are forced to more vigorous life, of which an increased formation of creatin is the result.

Quite in accordance with the investigations and arguments of FOLIN, v. HOOGENHUYZE and VERPLOEGH also found that though the excretion of urea increases and decreases with the resorption of proteids, the excretion of creatinin is not directly dependent on it. There is dependence in so far that with total privation of food, the activity of the organs becomes as small as possible and that then with the intensity of the symptoms of life the secretion of creatinin becomes extraordinarily small. In connection with this a statement made on the last day of the hunger-period of TOSCA is worth mentioning. June the 25th she took milk and eggs in the evening after ten o'clock. The urine which was collected the following morning at 10 'clock contained 0.375 gr. creatinin, more than double the quantity which was excreted by her in the last days in that same period. This sudden increase can certainly not be put to the account of the food as such, as is shown by the very slight increase of the excretion of nitrogen in the same period, but must be attributed to the stimulation which the whole organism suffered by the putting into action of the digestive organs after such a long rest.

NOËL PATON investigated a short time ago with the aid of FOLIN's method the excretion of creatinin of a dog which was fed with oatmeal and milk and moreover on one day with 5 eggs and which got no food at all on other days [1]). According to the author the results seem to indicate that in the dog there is a relationship between the production of creatinin and the intake of nitrogen.

The secretion of creatinin shows a somewhat too large irregularity in the communicated series to admit the making of conclusions. But if the impression of the author is right, there may be thought here also of a stimulating effect of the food on the whole organism.

Just as FOLIN, VAN HOOGENHUYZE and VERPLOEGH have observed not unimportant individual varieties in the excretion of creatinin with mixed food. Without doubt the quantity of meat which one is used to take, influences it. But with persons living pretty well under the same circumstances the difference seems to be less great when the weight of the body is considered. In 5 students a secretion of 26, 26.9, 27.4, 29,4 and 31,5 mgr. creatinin pro bodily weight of one Kgr. was found in 24 hours.

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[1]) Journal of Physiol. Vol. XXXIII, p. 1.

VAN HOOGENHUYZE and VERPLOEGH have also examined the urine of some sucklings. Always creatinin could be shown, more distinctly with the reaction of JAFFÉ than with that of WEYL. On account of the small concentration and the trifling quantities of urine which could be collected an accurate colorimetric determination was not possible. In four cases however a sufficient quantity of urine (15—60 cc.) was obtained, to admit at least of a somewhat reliable determination. In 10 cc. urine which was diluted to 50 cc. after having been mixed with picric acid and caustic soda, there was found:

I	child	8 days	old,	1.11	mgr. creatinine
II	„	32 „	„	0.91	„ „
III	„	2 months	„	0.41	„ „
IV	„	2 „	„	1.7	„ „

It is remarkable that in case III which concerned a weak child which was fed exclusively on cowmilk, the quantity of creatinin was so much smaller than in the three other children who were all strong and brought up by human-milk.

The above mentioned proves, as it appears to me, that the method of FOLIN is an acquisition of importance of which may be expected that it will aid in penetrating deeper than before into the knowledge of metabolism.

Physics. — "*On the theory of reflection of light by imperfectly transparent bodies.*" By Prof. R. SISSINGH. (Communicated by Prof. H. A. LORENTZ).

1. The laws of metallic reflection have been derived first by CAUCHY [1]), later by KETTELER [2]) and VOIGT [3]), while LORENTZ [4]) has developed them from the electromagnetic theory of light. By different

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[1]) CAUCHY, Compt. Rend. **2**, 427, 1836; **8**, 553, 658, 1839; **9**, 726, 1839; **26**, 86, 1848; Journ. de Liouv, (1), **7**, 338, 1839. CAUCHY gives only general remarks on the way followed by him. Derivations of the results have been given, inter alia by BEER, Pogg. Ann. **92**, 402, 1854; ETTINGSHAUSEN, Sitzungs-Ber. Akad. Wien, **4**, 369, 1855; EISENLOHR, Pogg. Ann., **104**, 368, 1858; LUNDQUIST, Pogg. Ann., **152**, 398, 1874.

[2]) Pogg. Ann., **160**, 466, 1877; Wied. Ann., **1**, 225. 1877; **3**, 95, 1878; **22**, 204, 1884. KETTELER has, also in consequence of VOIGT's observations, modified his developments, and given a final form to them in the "Theoretische Optik", 1885.

[3]) Wied. Ann., **23**, 104, 554, 1884; **31**, 233, 1887; **43**, 410, 1891.

[4]) On the theory of reflection and refraction of light, 1875; SCHLÖMILCH's Zeitschr. f. Math. u. Physik, **23**, 196, 1878.

ways these investigators arrive at exactly the same results. The relation inter se of the mechanic theories has been elucidated by DRUDE [1]). In 1892 LORENTZ [2]) derived the laws of the refraction of light by metal prisms, which had already been given by VOIGT [3]) and DRUDE [4]), from a few simple principles. Concerning the nature of the vibrations of light no special hypothesis is introduced. This investigation of LORENTZ enables us to develop the theory of metallic reflection in a simple way.

2. The simplest disturbance in a metal is that represented by:

$$A\, e^{-px} \sin (ct - qx + s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this x is the distance from the bounding plane of the metal. This disturbance is caused when light falls perpendicularly on the metal. Here we meet with the particularity, that the planes of equal phase determined by the goniometric factor of (1) coincide with these of equal amplitude which follow from the exponential factor. From the assumption that the metal is isotropic and the deviation from the condition of equilibrium in the light disturbance is a vector determined by homogeneous linear differential equations, LORENTZ derives, what other disturbances are possible in the metal. Assume that the bounding plane of the metal is the YZ-plane, and that the plane wave-fronts are perpendicular to the XZ-plane. Then a disturbance is possible, represented by:

$$A\, e^{-P l_1} \sin (c\, t - Q\, l_2 - s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if

$$P^2 - Q^2 = p^2 - q^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$PQ \cos (\alpha_1 - \alpha_2) = p\, q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

are satisfied.

The planes of equal amplitude and phase are given by $l_1 = \text{const.}$, $l_2 = \text{const.}$ In this l_1 is the distance to the plane in which the amplitude is A, and l_2 that to the plane in which the phase has the value s. α_1 and α_2 are the angles of the normals of the planes of equal amplitude and phase with the X-axis.

3. From (3) and (4) the principal equations for the propagation of light in metals may be immediately obtained. If light penetrates from the surroundings into the metal, then the planes of equal amplitude are parallel to the bounding plane. The exponential factor

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[1]) Göttinger Nachrichten 1892, 366, 393.
[2]) Wied. Ann., **46**, 244. 1892.
[3]) Wied. Ann., **24**, 144, 1885.
[4]) Wied. Ann., **42**, 666, 1891.

in (2) passes into e^{-Px} and $\alpha_1 = 0$. In this case α_2 may be called the angle of refraction in agreement with what takes place for perfectly transparent bodies. Denote it by α, then (4) passes into:

$$PQ \cos \alpha = pq. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let us now put $P = 2\pi k : \lambda$, where λ is the wave length in the air, and k the coefficient of absorption. In (2) we put $Q = 2\pi : \lambda_1$, where λ_1 represents the wave length in the metal. Be $\lambda : \lambda_1 = n$, then we may call the index of refraction of the metal n, in agreement with what happens in transparent bodies. In the same way $Q = 2\pi n : \lambda$. Let us call the values of l and n, when the light propagates in the metal perpendicularly to the bounding plane k_0 and n_0. Then in (1) $p = 2\pi k_0 : \lambda$, $q = 2\pi n_0 : \lambda$.

Introducing these values into (3) and (5), we get:

$$k^2 - n^2 = k_0^2 - n_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$kn \cos \alpha = k_0 n_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In order to bring our formulae for the disturbance in the metal at the bounding plane in harmony with those for the disturbance in the air, we must put $\sin i : \sin \alpha = \lambda : \lambda_1 = n$ or $\sin i = n \sin \alpha$, when i is the angle of incidence. It follows from (6) and (7) that both the index of refraction and the coefficient of absorption depend on the direction of propagation, i. e. on the direction of the normal of the planes with equal phase.

If (7) is written in the form:

$$k^2 n^2 \cos^2 \alpha = k^2 (n^2 - \sin^2 i) = k_0^2 n_0^2 \quad . \quad . \quad . \quad . \quad (8)$$

it follows from (6) and (8) that:

$$2n^2 = -k_0^2 + n_0^2 + \sin^2 i + \sqrt{(k_0^2 - n_0^2 + \sin^2 i)^2 + 4n_0^2 k_0^2} \quad . \quad . \quad (9)$$

$$2k^2 = \quad k_0^2 - n_0^2 + \sin^2 i + \sqrt{(k_0^2 - n_0^2 + \sin^2 i)^2 + 4n_0^2 k_0^2} \quad . \quad . \quad (10)$$

They denote in what way k and n depend on the angle which the direction of the propagation of the disturbance falling on the metal forms with the normal to the bounding plane [1]).

For an opaque mirror of silver deposited on glass by a chemical

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[1]) KETTELER was the first to derive these equations, (see inter alia Pogg. Ann., **160**, 408, 1877) which, of course, also occur in VOIGT's theory. VOIGT puts the quantity corresponding to P equal to $2\pi k : \lambda_1$, so that VOIGT's nk corresponds to the coefficient of absorption k introduced here. It is not correct that CAUCHY already gave these equations, as DRUDE observes (Wied. Ann. **35**, 515, 1888). They have not been given explicitly in this theory. This appears, indeed, from the fact that BEER (Pogg. Ann., **92**, 412, 1854) substitutes other relations for them, which are not correct. Derivations of the principal equations were given by WERNICKE (Pogg. Ann. **159**, 226, 1876) and KETTELER, Pogg. Ann. **160**, 468, 1877. See also KETTELER, Wied. Ann., **49**, 512, 1893 and Theoretische Optik, p. 198, § 85, Zur Geschichte der Hauptgleichungen.

way, and very firmly attached to the glass, the values $I = 72°34'.8$, $H = 42\ 21'.7$ were found for principal angle of incidence I and principal azimuth H[1]) (H being the angle which the plane of polarization of the reflected light, being restored by compensation to plane polarization makes with the plane of incidence) from which follows for:

$i = 0°$	20°	40°	60°	80°	90°
$n = 0{,}295$	0,450	0,800	0,928	0,990	1,03
$k = 2{,}88$	2,90	2,95	3,01	3,04	3,05

In the same way I found for a steel mirror[2]), $I = 77°23'.5$, $H = 26°34'$, so that for:

$i = 0°$	$i = I$	$i = 90°$
$n = 2{,}684$	2,794	2,799
$k = 3{,}404$	3,491	3,496

As follows from (9) and (10), k and n increase with the angle of incidence i. From (8) follows, that always $n^2 > \sin^2 i$. Media which absorb the light, can never reflect the light totally.

4. Normal to the planes of equal amplitude the amplitude decreases in ratio $1 : e^{-1}$ over a distance $\lambda : 2\pi k$. In the planes of equal phase the points whose amplitudes stand in the same ratio, lie at a distance $\lambda : 2\pi k \sin(\alpha_1 - \alpha_2)$.

According to (6) and (7) n depends on k. The velocity of propagation depends therefore on the way, in which the amplitude in a plane of equal phase varies. If $\alpha = 0$, it follows from (6) and (7) that $k = k_0$, $n = n_0$. The planes of equal phase and amplitude can therefore only coincide with a propagation normal to the bounding plane. If this took place in every direction, k would be zero according to (8), so the substance would have to be perfectly transparent.

When the planes of equal phase and amplitude are normal to each other, $\alpha = 90°$. For light that penetrates into the metal from outside, the planes of equal amplitude are parallel to the bounding plane, so for $\alpha = 90°$ those of equal phase are perpendicular to it. The propagation then takes place parallel to the bounding plane. This is in harmony with what follows from (7) and (8). According to (7) $k_0 n_0 = 0$ for $\alpha = 90°$ and so according to (8) either $k = 0$ or $n = \sin i$. The first case leads us back to perfectly transparent media. For $n = \sin i$ there is total reflection. This however, can only be the case with light absorbing media, if $k_0 n_0 = 0$ or, as $n_0 > 0$, if

[1]) Sissingh, Thesis for the doctorate, p. 88, 1885, Arch. Néerl., **20**, 207, 1886.

[2]) Sissingh, Verh. Akad. v. Wetensch., Amsterdam, deel 28, 1890; Wied. Ann., **42**, 132, 1891.

$k_0 = 0$. So the coefficient of absorption of the medium normal to the bounding plane had to be 0. For metals this is not the case, so that no total reflection can occur there, as has been observed above.

It is well known that with total reflection on perfectly transparent media the planes of equal phase and amplitude are normal to each other for the disturbance in the second medium which is propagated parallel to the bounding plane. VOIGT showed, that this case also occurs for a disturbance, which leaves a prism of a substance which absorbs light, when plane waves fall on it and the dimensions of the prism are large with respect to the wave length [1]).

From (6) and (7) we may derive $(k_0^2 - n_0^2) \cos \alpha = n_0 k_0 \left(\frac{k}{n} - \frac{n}{k}\right)$. From this follows, that according as $k : n$ increases, α differs more from $\pi : 2$, with which we have got back a result of VOIGT's [2]).

5. EISENLOHR [3]) showed, that by the introduction of a complex index of refraction, we arrive at CAUCHY's results for metallic reflection.

In the following way it may be shown that for metals a complex quantity corresponds to the index of refraction of transparent bodies. With observance of the conditions (3) and (4), (2) is a possible disturbance. In this l_1 and l_2 are the distances from the point for which (2) holds, to the plane of equal amplitude, in which the amplitude is A and the plane of equal phase, in which the phase is s. We may also write for (2):

$$Ae^{-p_1 x - p_2 z} \sin (ct - q_1 x - q_2 z - s) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

because the planes of equal phase and amplitude are normal to the XZ-plane. The normals from the point x,z on the two above mentioned of these planes are respectively $(p_1 x + p_2 z) : \sqrt{p_1^2 + p_2^2}$ and $(q_1 x + q_2 z) : \sqrt{q_1^2 + q_2^2}$, so that $P = \sqrt{p_1^2 + p_2^2}$, $Q = \sqrt{q_1^2 + q_2^2}$.

In the same way as (11) a possible disturbance is also:

$$Ae^{-p_1 x - p_2 z} \cos (ct - q_1 x - q_2 z - s).$$

The differential equations, which are supposed homogeneous and linear, are therefore also satisfied by:

$$Ae^{-p_1 x - p_2 z} \{\cos (ct - q_1 x - q_2 z - s) \pm i \sin (ct - q_1 x - q_2 z - s)\}$$

or by

$$Ae^{\pm \{(ct - x(q_1 \mp \psi_1) - z(q_2 \mp \psi_2) - s\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

For a perfectly transparent medium $p_1 = p_2 = 0$. The velocity of propagation is then $v = c : \sqrt{q_1^2 + q_2^2}$, or c being $c = 2\pi : T$,

[1]) Wied. Ann., **24**, 153, 1885.
[2]) Wied. Ann, **24**, 150, 1885.
[3]) Pogg. Ann., 104, 368, 1858.

$v = 2\pi : T\sqrt{q_1^2+q_2^2}$. Let the velocity of light in the air be V, the index of refraction of the perfectly transparent medium n, then:

$$n^2 = V^2T^2(q_1^2+q_2^2) : 4\pi^2$$

From (12) follows, that for a metal $q_1 \mp \iota p_1$ occurs instead of q_1 and the quantity $q_2 \mp \iota p_2$ for q_2. Let n_m be the quantity, which for a metal corresponds to the index of refraction n of the perfectly transparent medium, then

$$n_m^2 = \frac{V^2T^2}{4\pi^2}\{-p_1^2+q_1^2-p_2^2+q_2^2 \mp 2\iota(p_1q_1+p_2q_2)\}.$$

The cosines of the angles formed by the normals of the planes of equal amplitude and phase with the X and Z-axis, are respectively:

$$p_1 : \sqrt{p_1^2+p_2^2}\ ,\ p_2 : \sqrt{p_1^2+p_2^2} \text{ and } q_1 : \sqrt{q_1^2+q_2^2}\ ,\ q_2 : \sqrt{q_1^2+q_2^2}$$

With observance of the above given values of P and Q and introduction of the angle α between the planes of equal phase and amplitude $p_1q_1+p_2q_2 = PQ\cos\alpha$. Thus:

$$n_m^2 = \frac{V^2T^2}{4\pi^2}(-P^2+Q^2 \mp 2\iota PQ\cos\alpha),$$

or according to (3) and (5):

$$n_m^2 = \frac{V^2T^2}{4\pi^2}\{-p^2+q^2 \mp 2\iota pq\}.$$

Hence the so-called complex index of refraction of a metal is $n_m = (\iota \pm q)\frac{VT}{2\pi}$. Let λ_0 be the wave length in the metal for light entering normally, then according to (1) $q = 2\pi : \lambda_0 = 2\pi n_0 : \lambda$ and $p = 2\pi k_0 : \lambda$, so $n_m = n_0 \mp \iota k_0$.

6. It follows from what precedes, that in accordance with EISENLOHR [1]) we can deduce the expressions determining the amplitudes for the metallic reflection from those for the reflection on transparent bodies, if we replace n by $n_0 \mp \iota k_0$. Let the incident beam of light have the intensity 1 and let it be polarized in the plane of incidence. The reflected disturbance may be represented by the real part of $\frac{\sin(i-r)}{\sin(i+r)} e^{\pm\iota(ct+\chi)}$. Here $\sin r = \sin i : n$. Put $n = n_0 \mp \iota k_0$, then $\frac{\sin(i-r)}{\sin(i+r)}$ passes into $Ae^{\pm\iota B}$. The disturbance reflected by metals is the real part of $Ae^{\pm\iota(ct+\chi-B)}$, in which A is the amplitude and B the difference of phase with the incident ray. In this way we arrive at the well known expressions for the metallic reflection. I may be

[1]) Cf. also LORENTZ, Theorie der Terugkaatsing en Breking, p. 163, SCHLÖMILCH's Zeitschr., **23**, 206, 1878.

allowed to place them here side by side, after which I shall give some expressions which enable us to determine the optical constants of a metal from the quantities measured, and also some approximative formulae for the calculation of the principal angle of incidence I and principal azimuth H from n_0 and k_0.

Light polarized // plane of incidence.

	Reflection by transparent bodies	Reflection by metals
	Intensity	
Incident light	Reflected light	
1	$\frac{\sin^2(i-r)}{\sin^2(i+r)}$	$R_p^2 = \frac{(\cos i - \sqrt{n^2-\sin^2 i})^2+k^2}{(\cos i+\sqrt{n^2-\sin^2 i})^2+k^2}$
	Difference of phase with incident beam	
	$180°$	$tg\varphi_p = -\frac{2k\cos i}{1-n^2-k^2}$

Light polarized $\perp$ plane of incidence

	Intensity	
1	$\frac{tg^2(i-r)}{tg^2(i-r)}$	$R_l^2 = \frac{n^2\cos^2(i-\alpha)+k^2\cos^2 i}{n^2\cos^2(i+\alpha)+k^2\cos^2 i} R_p^2$
	Difference of phase with incident beam	
	$0°$ for $0° < i < I$	$tg\varphi = C$
	$180°$ „ $I < i < 90°$	

$$C = \frac{2k(k^2+n^2\cos^2\alpha-\sin^2 i)\cos i}{(k^2-n^2\cos^2\alpha-\sin^2 i)\cos^2 i - n^2\cos^2\alpha\cos 2i - k^2}$$

From this follows also:

$$tg(\varphi_l - \varphi_p) = \frac{2k \sin i\, tg\, i}{n^2\cos^2\alpha - tg^2 i \sin^2 i + k^2},$$

n and k apply to the disturbance in the metal, arising from plane waves, which fall on the metal with an angle of incidence i. The angle of refraction α is determined by $Sin\,\alpha = Sin\, i : n$.

7. The expressions obtained are in perfect harmony with those of CAUCHY. First the relations of § 6 may be brought to the same form as these. For this purpose we put in accordance with BEER [1]):

$$n\cos\alpha = \sqrt{n^2 - Sin^2 i} = U\cos u, \quad k = U\, Sin\, u \quad . \quad . \quad (13)$$

Substituting this we get

$$R_p^2 = \frac{\cos^2 i + U^2 - 2U\cos i\cos u}{\cos^2 i + U^2 + 2U\cos i\cos u}.$$

Put

1) BEER, Pogg. Ann. **92**, 413, 1854.

$$tg\, f = \frac{cos^2 i + U^2}{2\, U\, cos\, i\, cos\, u},$$

then

$$R_p^2 = tg\left(f - \frac{\pi}{4}\right).$$

From the value of f follows also

$$cot\, f = cos\, u\, sin\left(2\, Bg\, tg\, \frac{cos\, i}{U}\right) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Further we get:

$$tg\, \varphi_p = sin\, u\, tg\left(2\, Bg\, tg\, \frac{cos\, i}{U}\right).$$

Be $R_l : R_p = tg\, h$, then

$$tg^2\, h = \frac{n^2\, cos^2\,(i-\alpha) + k^2\, cos^2\, i}{n^2\, cos^2\,(i+\alpha) + k^2\, cos^2\, i}.$$

In the corresponding expression of CAUCHY the value of $Cos\, 2\, h$ is given. From the value of $tg^2\, h$ follows, as

$$cos\, 2h = (1 - tg\, 2h) : (1 + tg\, 2h)\ , \quad cos\, 2h = \frac{2n\, cos\, \alpha\, sin^2\, i\, cos\, i}{(n^2 cos^2 \alpha + k^2) cos^2 i + Sin^4 i}.$$

According to (13) this passes into

$$cos\, 2h = cos\, u\, sin\left(2\, Bg\, tg\, \frac{sin\, i\, tg\, i}{U}\right). \quad . \quad . \quad . \quad . \quad (15)$$

In the same way becomes

$$tg\,(\varphi_l - \varphi_p) = sin\, u\, \frac{2\, U\, sin\, i\, tg\, i}{U^2 - sin^2\, tg^2\, i} = sin\, u\, tg\left(2\, Bg\, tg\, \frac{sin\, i\, tg\, i}{U}\right) . \quad (16)$$

The expressions (14), (15), (16) have the same form as the corresponding ones of CAUCHY, only according to LORENTZ's notation $\sigma\varrho$ stands for U, the angle $\tau + \omega$ for u. [1])

Just as from i, U and u the quantities R_p, R_l, φ_p, and φ_l may be derived, which determine the reflected beam of light, U and u may be calculated from i and two of these four quantities. U and u depend therefore in exactly the same way on the angle of incidence and the optical properties of the metal as CAUCHY's corresponding quantities $c\varrho$ and $\tau + \omega$. At the same time it appears that two constant quantities suffice for the determination of the optical behaviour of metals. They are here n_0 and k_0, which have a definite physical meaning, with CAUCHY σ and τ, whose meaning is not so obvious. Whatever

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[1]) LORENTZ, Theorie der terugkaatsing en breking. p. 166. According to EISENLOHR's notation (loc cit. p. 369, 370) $U = c\,\vartheta$, $u = \varepsilon + u$. As R_p and $R_l : R_p$, φ_p and $\varphi_l - \varphi_p$ have the same form as CAUCHY's equation, this holds also for R_l and φ_l,

system of two determining quantities is chosen, however, the amplitudes and phases of the two components of the reflected light polarized in and normal to the plane of incidence, whether calculated in one way or in the other, will have the same values. The two systems of formulae are therefore identical.

CAUCHY [1]) calls the so-called complex index of refraction $\sigma e^{\iota\tau}$. This quantity being represented here by $n_0 + \iota k_0$, $\sigma \cos\tau = n_0$, $\sigma \sin\tau = k_0$. The auxiliary quantities ϱ and ω have been introduced by CAUCHY for the determination of the so-called imaginary angle of refraction r, determined by $\sin r = \sin i : (n_0 + \iota k_0)$ [2]). In order to express ϱ and ω in the quantities used above, it may be observed that:

$$\cos^2 r \sin^2 i : \sin^2 r = \cos^2 r (n_0 + \iota k_0)^2 = (n_0 + \iota k_0)^2 - \sin^2 i$$

or with the aid of (6) and (7):

$$\cot r \sin i = n \cos\alpha + \iota k.$$

As CAUCHY gives:

$\cos r = \varrho e^{\iota\omega}$ [1]) and $n_0 + \iota k_0 = \sigma e^{\iota\tau}$, $n\cos\alpha + \iota k$ is equal to $\varrho\tau e^{\iota(\tau+\omega)}$ or

$$n \cos\alpha = U \cos u = \varrho\sigma \cos(\tau + \omega) \quad . \quad . \quad . \quad . \quad (17)$$

$$k = U \sin u = \varrho\sigma \sin(\tau + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The equations (17) and (18) allow us to deduce our auxiliary quantities from those of CAUCHY and reversely [3]).

8. According to § 7:

$$\cos 2h = \cos u \frac{2U \sin i \operatorname{tg} i}{U^2 + \sin^2 i \operatorname{tg}^2 i}, \operatorname{tg}(\varphi_l - \varphi_p) = \sin u \frac{2U \sin i \operatorname{tg} i}{U^2 - \sin^2 i \operatorname{tg}^2 i}.$$

These two equations may serve to determine U and u, and from this the optical constants n_0 and k_0 with the aid of (13), (6) and (7).

From the values of $\cos 2h$ and $\operatorname{tg}(\varphi_l - \varphi_p)$ follows:

$$\sin 2h \cos(\varphi_l - \varphi_p) \frac{U^2 - \sin^2 i \operatorname{tg}^2 i}{U^2 + \sin^2 i \operatorname{tg}^2 i}$$

or

$$1 - \sin 2h \cos(\varphi_l - \varphi_p) = \frac{2 \sin^2 i \operatorname{tg}^2 i}{U^2 + \sin^2 i \operatorname{tg}^2 i} \quad . \quad . \quad . \quad (19)$$

From (19) and the value of $\cos 2h$ follows:

$$U \cos u = \frac{\sin i \operatorname{tg} i \cos 2h}{1 - \cos(\varphi_l - \varphi_p) \sin 2h} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

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[1]) LORENTZ, l.c. Theorie der terugkaatsing en breking, p. 164, SCHLÖMILCH's Zeitschr., **23**, pg. 206. EISENLOHR, p. 369.

[2]) See note 1 on the preceding page.

[3]) Cf. also KETTELER, Wied. Ann. **1**, 242, 1877; **22**, 212, 1884. Formulae for the calculation of ϱ and ω are given by LORENTZ, Theorie der Terugkaatsing en Breking, p. 164, 165, EISENLOHR, Pogg. Ann., **104**, 370, 1858.

Further

$$\sin 2h \sin(\varphi_l - \varphi_p) = \frac{2U \sin u \sin i \operatorname{tg} i}{\sqrt{U^2 + \sin^2 i \operatorname{tg}^2 i}}.$$

From this and from the value of $\operatorname{tg}(\varphi_l - \varphi_p)$ follows

$$U \sin u = \frac{\sin i \operatorname{tg} i \sin(\varphi_l - \varphi_p) \sin 2h}{1 - \cos(\varphi_l - \varphi_p) \sin 2h} \quad . \quad . \quad . \quad . \quad (21)\ ^{1})$$

So from the restored azimuth h and the difference of phase $\varphi_l - \varphi_p$ at an arbitrary angle, $U \cos u$ and $U \sin u$ or $n \cos \alpha$ and k are to be derived for that angle. As $n \cos \alpha = \sqrt{n^2 - \sin^2 i}$, we get afterwards n_0 and k_0 with the aid of (6) and (7). By means of them we can calculate $\varphi_l - \varphi_p$ and h for every angle.

9. As a rule we introduce the principal angle of incidence I, for which $\varphi_l - \varphi_p = \pi : 2$. The restored azimuth at this angle is called the principal azimuth H. As well from (20) and (21), as from 15) and (16) we may derive, when we add the index I to the values of all the quantities for the principal angle of incidence:

$$U_I = \sin I \operatorname{tg} I \quad , \quad \cos u_I = \cos 2H \quad . \quad . \quad . \quad . \quad (22)$$

According to (13)

$$k_I = U_I \sin u_I = \sin I \operatorname{tg} I \sin 2H. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$(n^2 \cos^2 \alpha)_I = n_I^2 - \sin^2 I = \sin^2 I \operatorname{tg}^2 I \cos^2 2H$$

or

$$n_I^2 = \operatorname{tg}^2 I (1 - \sin^2 I \sin^2 2H) \quad . \quad . \quad . \quad . \quad . \quad (24)$$

We may also write (24):

$$n_I^2 + k_I^2 = \operatorname{tg}^2 I\ ^{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The optical constants n_0 and k_0 are obtained from:

$$n_0^2 - k_0^2 = n_I^2 - k_I^2 = \operatorname{tg}^2 I (1 - 2 \sin^2 I \sin^2 2H)$$

or

$$n_0 k_0 = (n \cos \alpha)_I k_I = \tfrac{1}{2} \sin^2 I \operatorname{tg}^2 I \sin 4H \quad . \quad . \quad . \quad (26)$$

10. When n_0 and k_0 have been given, we find by elimination of n_I and k_I from the two first members of the two equations (26) and (25) an equation of the sixth degree for the determination of $\operatorname{tg} I$. There may be given also approximating formulae for the determination of I and H from n_0 and k_0. From $n_I^2 - k_I^2 = n_0^2 - k_0^2$ and $k_I \sqrt{n_I^2 - \sin^2 I} = n_0 k_0$ follows

$$2n_I^2 = \sin^2 I + n_0^2 - k_0^2 + \sqrt{(\sin^2 I - n_0^2 - k_0^2)^2 + 4k_0^2 \sin^2 I}$$

Substituting this in $n_I^2 + k_I^2 = \operatorname{tg}^2 I$, we get:

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[1]) This equation was already given by KETTELER, Wied. Ann,. **1**, 241, 1877.

[2]) KETTELER calls this equation an analogon of the law of BREWSTER, Wied. Ann **1**, 242, 1877.

$$\sin^4 I + 2\sin^2 I(k_0^2 - n_0^2) + (k_0^2 + n_0^2)^2 = \sin^4 I \operatorname{tg}^4 I \quad . \quad . \quad (27)$$

With metals $n_0^2 + k_0^2$ is comparatively large compared to the two first terms of the first member of (27). By approximation we get therefore:

$$\sin^2 I \operatorname{tg}^2 I = k_0^2 + n_0^2,$$

from which follows with the same degree of approximation $\sin^2 I = 1 - \frac{1}{k_0^2 + n_0^2}$.

Introducing this in (27), we get:

$$\sin I \operatorname{tg} I = \sqrt{k_0^2 + n_0^2}\left\{1 + \frac{1}{4}\frac{2(k_0^2 - n_0^2) - 1}{(n_0^2 + k_0^2)^2}\right\} \quad . \quad . \quad . \quad (28)$$

In the following way we get an approximate value for H. From (23) and (24) follows:

$$n_I^2 - k_I^2 = n_0^2 - k_0^2 = \sin^2 I + \sin^2 I \operatorname{tg}^2 I \cos 4H,$$

so

$$\cos 4H = \frac{n_0^2 - k_0^2 - \sin^2 I}{\sin^2 I \operatorname{tg} I}$$

From this follows, as $\operatorname{tg}^2 2H = \frac{1 - \cos 4H}{1 + \cos 4H}$, after substitution of the approximate value

$$\sin^2 I \operatorname{tg}^2 I = (n_0^2 + k_0^2)\left\{1 + \sin^2 I \frac{k_0^2 - n_0^2}{k_0^2 + n_0^2}\right\}$$

which follows from (27),

$$\operatorname{tg} 2H = \frac{k_0}{n_0}\left\{1 + \sin^2 I \frac{1}{n_0^2 + k_0^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (29)\ ^1)$$

11. Finally it may be observed that the relations hold for any value of k. The reflection on perfectly transparent bodies is therefore a limiting case for the metallic reflection. ²)

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(This communication will not be published in these Proceedings).

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¹) Corresponding approximate formulae were given by DRUDE in WINKELMANN, Physik II. 1, p. 823, 824.

²) Cf. VOIGT, Wied. Ann., **24**, 146, 147, 1885.

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